Risk Taking and Fiscal Smoothing with Sovereign Wealth Funds in Advanced Economies

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Abstract

We analyse the interaction between fiscal policy and portfolio management for the government of an advanced economy with a sovereign-wealth fund (SWF). We consider risk taking simultaneously with the use of SWF draws in fiscal policy. Assuming non-expected utility preferences allows us to distinguish between policy makers' tolerance of intended versus stochastic variations in SWF draws. The desire for smoothness in taxes and public services translates into forward as well as backward smoothing of SWF draws. Backward smoothing translates into risk aversion and may even call for pro-cyclical rebalancing. Future interest rates are associated with interest-rate risk. We show that this risk may lead to higher optimal risk taking. We further show that policy makers can use the draw rates from the SWF to smooth over time variation in risk-free rates.

JEL classification: G11; G23

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1. Introduction

Although much has been written on sovereign debt management as a part of fiscal policy, much less has been done on the challenges facing fiscal policy makers in the presence of a sizeable sovereign wealth fund (SWF). Although a net asset position may seem obviously preferable to net debt, the simultaneous tasks of investing the fund and using it as a source of fiscal revenue is far from trivial. The recent proliferation of such funds furthermore means that analyses of the issues that then arise has much more than academic interest. Some of the contributions made so far have focused on SWFs in emerging economies. Although this may be natural considering the relative prevalence of SWFs in such economies, we find that fiscal policy making with a SWF in an advanced-market economy also raises new issues significant enough to deserve special analysis. Our work has been inspired by the issues facing the Norwegian government, whose SWF has reached a magnitude of about $1 trillion, and one seventh of the government's budget is financed by draws on the fund[1]. The issues that arise are general, however, and not limited to the Norwegian case.

The discovery and exploitation of non-renewable natural resources typically give rise to substantial, though temporary public revenues. In order to avoid the well-known resource curse, often referred to as Dutch disease (van Wijnbergen, 1984; Corden, 1984, and many others), a number of countries in this situation have established SWFs to avoid overheating and to preserve the wealth for future generations. Examples besides Norway include Saudi Arabia and other states in the Persian Gulf, Chile, and New Zealand. Furthermore, the Asian financial crisis in 1998 motivated several countries to build large foreign-exchange reserves. China has converted most of its reserves into two regular SWFs, independent of FX management; and Singapore manages no less than three funds to safeguard future pensions and as general buffers. The fiscal surpluses in the United States in the late 1990s reportedly got some influential people starting to think about the possibility of an SWF even there (see Greenspan, 2007 pp. 217-218); and the Swedish AP Funds are an example of a significant advanced-country SWF established without special resource revenues.

[1] One of us participated in two government-appointed Commissions related to the investment strategy and the fiscal rule associated with this SWF, Norwegian Ministry of Finance (2015, 2016).

An important difference between fiscal policy with net debt and one with a SWF is that portfolio choice is much more important with an SWF. Furthermore, whereas high risk taking normally means higher expected returns, this prospect must be weighed against the need for stability in tax rates and government services. Conversely, caution in terms of risk taking may mean that the proceeds of the fund will be less able to support government spending. Thus, once an SWF has been established, policy makers have to decide on some important issues: First, how much risk to take in the asset portfolio, and second, how much to draw from the fund to support current spending. A third decision will be equally important, namely, how to distribute the draw from the fund over time. The goal of this paper is to address all three of these issues and how they fit into the overall fiscal policy framework. The issues are related and we address them simultaneously.

Whereas policy makers may favour risky investments in order to harvest high risk premia, the value of risky investments may fluctuate significantly. At the same time, policy makers may not be at liberty to make correspondingly large changes in spending from one period to the next; and sound arguments call for taxes and public services to change smoothly over time, if at all (Barro, 1979). Furthermore, although risk premia may motivate high risk taking, policy makers appear to have low tolerance for non-stochastic, or planned, variation over time in tax rates and services and a strong desire to preserve value for future generations. This combination of attitudes cannot be reconciled within the standard expected utility framework. To encompass these stylised facts, we use non-expected utility preferences as proposed by Epstein and Zin (1991). These preferences allow us to distinguish between risk aversion and willingness to intertemporal substitution. Furthermore, we borrow the tools of habit formation (see e.g., Constantinides, 1990, Campbell and Cochrane, 1999) to include preferences for smooth changes in public services. We derive closed-form solutions for the portfolio selection problem. Interestingly, the preferences for planned variation over time do not affect the portfolio choice, i.e., the distribution between risky and non-risky assets. Although this insight has already been established in the literature going back to Svensson (1989) for the basic case of a constant riskless rate and no habit formation, we show that it holds also when these assumptions are relaxed.

Modelling policy makers' preferences for smoothness of tax rates and public services translate into a desire for stability in the regular draws that can be made on the fund to finance public spending. Although a full modelling of desire is possible, it will generally not lend itself to informative, closed-form solutions. As an introduction of the issue in the paper, we choose instead to approximate policy makers' preference as a case of habit formation. Under this assumption, we derive closed form solutions for how the preferences for maintaining the spending habit affect the spending rate. Because, in contrast, risk taking feeds volatility, we find habit-influenced preferences to have a profound effect on the portfolio selection problem. They reduce the short-term risk taking because a larger part of the investment portfolio must be used to safeguard the habit level of consumption.

A side effect of this smoothing of public spending and taxes is that the portfolio risk for the long run increases. We find that this increased portfolio risk spills over into public spending and can increase the long-run spending risk considerably. For short-horizon portfolio-selection problems, treasury STRIPS or other zero-coupon bonds can be good substitutes for risk-free investments. For portfolio-selection problems with long horizons, like the infinite horizons for many SWFs, "risk-free investments" are risky because interest rates fluctuate randomly. We address the effect of interest rate uncertainty on the optimal draw from the fund and on the portfolio-selection problem.

Both the portfolio choice and the spending decision are influenced by characteristics of the country owning the SWF. Relevant characteristics can be the country's stage of economic development and the government's budget, the fund's size relative to the global investment universe, and the fund's objectives. Empirical surveys, such as Johan, Knill, and Mauck (2013), Bernstein, Lerner, and Schoar (2013), and Dreassi, Miani, and Paltrienieri (2017) confirm this diversity, as does the literature survey by Alhashed (2015). Various papers have also sought to study these issues from a normative perspective. van den Bremer, van der Ploeg, and Wills focus on the interaction between the financial portfolio and the value of the natural resources that fund the financial portfolio. Although we find this issue important, we bypass it in our study, which then can be best interpreted as an analysis of SWF management and spending once the natural resource has been depleted. van der Ploeg and Venables (2011) focus on the case of developing economies, for whom limited access to global financial markets may present an argument for investing a disproportionate part of the fund domestically. Carroll and Jeanne (2009) and Sá and Viani (2013) follow the same vein by focusing on the effects on global balances and exchange rates of SWF development in emerging economies. Guerra-Salas (2014) has compared the effects of fiscal responses, including SWF investing as well as domestic public investment to oil price changes in Mexico and Norway. In a slightly different context, Collier and Gunning (2005) have argued for using an oil windfall primarily to reduce domestic debt.

Although the focus on emerging economies may be natural in many relevant cases, we believe that the challenges related to the investment and use of sovereign wealth funds are significant for developed-economy governments as well. Furhtermore, whereas small SWFs, such as the Chilean copper fund, mainly serve as buffers against temporary budget shortfalls, the financial returns of larger funds provide the government in question with a regular source of annual revenue over and above the normal tax system. Decisions about how the fund should be invested then resemble those in the classical analyses by Phelps (1962), Samuelson (1969), and Merton (1969) of an individual's optimal spending and portfolio allocation. The spending rules implied by these analyses resemble the well-known tenets of the permanent-income literature (e.g. Hall, 1978). However, the validity of the permanent-income rule rests on a number of simplifying assumptions that are not only unrealistic, but that also matter significantly to policy makers. First, it ignores risk and uncertainty, which obviously becomes important when an SWF is to be invested in the global financial markets. Second, it ignores the well-known and thoroughly analysed desirability of taxes and public services to change smoothly over time (Barro, 1979). SWF investing should thus be analysed in the broader framework of Asset Liability Management, as in Choudry (2007). And third, the substantial movements in risk-free interest rates in recent decades raise the question of how such movements should influence the rules for drawing from an SWF. Our paper addresses all of these concerns.

Our analysis focuses on portfolio selection, current spending, and planned variation in spending over time. The analysis is partial in the sense that it treats policy makers as a representative investor and consumer, but do not present a model for the underlying macro economy. Carrol and Jeanne (2009) and Sá and Viani (2013) analyse related topics to what we analyse. While they include a model for the macro economy, their asset-price dynamics and preference specification are simpler than ours are. Parts of our analysis follow from results in the existing literature, but the generalization to the case of non-expected utility is, to the best of our knowledge, novel for the cases of habit formation and time-varying risk-free rates. This generalization is particularly important for decisions regarding portfolio allocation for SWFs and the use of such funds as budget revenues. Yang (2015) includes habit formation and long-run risks in a non-expected utility framework. While his analysis is methodologically related to our analysis, his objective is not on portfolio choice and spending, but rather to analyse asset market phenomena from the macro-finance literature.

The paper is organised as follows. Section 2 uses Svensson's (1989) generalisation to Epstein-Zin preferences of the Merton model with a constant risk-free rate and no backward smoothing to gain some preliminary insights into the respective roles of risk aversion and intertemporal substitution for decisions about portfolio allocation and use of the proceeds of an SWF. Section 3 studies the implications of backward smoothing of tax rates and public services based on a generalisation of the model of Constantinides (1990). Such smoothing turns out to have implications for the normal rebalancing of the fund as well as its long-term performance, which we consider in Section 4. Section 5 extends the analysis from section 2 to the case of time variation in the risk-free rate. Section 6 presents our conclusions and some plans for further research.

2. Risk aversion vs intertemporal substitution

We start by considering the different roles of risk aversion and intertemporal substitution for the owner of an SWF. For this purpose, we use the Epstein-Zin formulation of non-expected utility, which in discrete time can be expressed by the following value function:

$$(1) \qquad V_t(W_t) = \max_{c_t,\alpha_t} \left\{ c_t^{1-\delta} + e^{-\rho} E_t [V_{t+1}(W_{t+1})^{1-\gamma}]^{(1-\delta)/(1-\gamma)} \right\}^{1/(1-\delta)}.$$

Here, ρ is the subjective discount rate, γ is the standard measure of relative risk aversion, and $\delta \equiv 1/\varepsilon$, where ε is the elasticity of intertemporal substitution. Thus, δ can be interpreted as a measure of aversion against non-stochastic or planned time variations of consumption c, which we interpret as the annual draw on the fund. W denotes wealth, i.e., the total value of the SWF portfolio. A fraction α of the fund's assets is invested in a risky asset (equity) and a fraction 1- α in a safe asset. As is well known, this formulation simplifies to the standard one of power expected utility if $\gamma = \delta$. However, we will not make this assumption because these two parameters serve substantially different functions in our present context, and policy makers' attitudes towards risk and non-stochastic time variation can be quite different.

Although decision making with non-expected utility is more easily analysed in discrete time, continuous-time analysis yields more informative solutions. For this reason, we carry out our analysis in continuous time, but leave most of the mathematical derivations to the Appendix.

As shown in Appendix A1, after transformation, taking the limit as the discrete time intervals approach zero, the value function (1) is equivalent to the Bellman equation

(2) $$0 = \max_{c(t),\alpha(t)} \left\{ c(t)^{1-\delta} - \rho U(W(t))^{(1-\delta)/(1-\gamma)} + \frac{1}{dt}[E_t dU(W(t))]^{(1-\delta)/(1-\gamma)} \right\},$$

where $U(W(t)) \equiv V(W(t))^{1-\gamma}$.

Equation (2) needs to be supplemented by a dynamic budget constraint. In this section, we assume that the risk-free return r is constant over time. In continuous-time notation, the flow budget constraint is given by

(3) $$dW(t) = [(r + \alpha(t)\mu)W(t) - c(t)]dt + \alpha(t)\sigma W(t) dB(t),$$

where $B(t)$ is a Wiener process. The risky asset has return $z(t) \sim NIID(r + \mu, \sigma^2)$, where μ is the time-invariant equity premium. Then, $r + \alpha(t)\mu$ is the expected portfolio return and $\alpha(t)^2\sigma^2$ its variance. This formulation ignores the possibility of stock-price mean reversion (e.g. Fama and French, 1988), as well as uncertainty about long-term trends (Bansal and Yaron, 2004, and Yang, 2015). We offer some informal comments on these issues below.

Proposition 1. *Under the assumptions given in (2) and (3), the optimal equity share α is constant and the optimal consumption is a constant share η of wealth, where*

(4a) $$\alpha = \frac{\mu}{\gamma\sigma^2} \equiv m$$

and

(4b) $$\eta = \varepsilon\rho + (1-\varepsilon)\left(r + m\mu - \frac{1}{2}\gamma m^2\sigma^2\right) = \varepsilon\rho + (1-\varepsilon)\left(r + \frac{1}{2}m\mu\right).$$

This proposition was originally proved by Svensson (1989). Our proof, which is somewhat different than his in order to facilitate our subsequent analysis, is provided in Appendix A1 and A2.

The solution for the equity share is identical to the one derived by Merton (1969) for the expected-utility case. Although known from previous research such as Svensson's, it may be worth noting that this part of Merton's results is not influenced by the double duty served by the risk-aversion parameter γ as the reciprocal of the elasticity of substitution in the expected-utility case[4].

Our main interest concerns the optimal draw on the fund. The draw rate in our case differs from the draw rate derived by Merton. Thus, while we have the same equity share as in the Merton model, going forward, the different draw rate makes the value of the investment portfolio, and thereby the *amount* invested in equity, different from the corresponding investments by the Merton investor. The draw rate is expressed as a linear combination—if $\varepsilon \leq 1$ as a weighted average—of what Giovanni and Weil (1989) and Campbell and Viceira (2002) refer to as a *myopic* and an *annuity* component, respectively. The myopic component may be large if the investor is impatient, so that ρ is large. For policy makers, this would be roughly equivalent to a desire to favour current generations over future ones. However, policy makers would then also need to be willing to plan for draws on the fund to vary non-stochastically over time, so that $\varepsilon > 0$.

In his seminal empirical study, Hall (1988) concludes that the intertemporal elasticity of substitution is likely to be small. This observation indicates that the income effect is more important than the substitution effect. Later studies, like Bansal and Yaron (2004) and Thimme (2016) have found larger values of ε, indicating that the substitution effect is important. However, our casual observations of policy-maker behaviour in advanced-economy countries with sovereign wealth funds suggest that concern about preserving the fund for future generations reveal, if anything, lower elasticities than those estimated for households. Acknowledging that ε is likely to be close to zero, we read SWF decision-makers as focusing mainly on the annuity component. This attitude would also be consistent with a desire for smoothness in the time-series behaviour of public services and tax rates, as

[4] It is also worth noting that the optimal equity share is not bounded above by unity. This is a feature of the model's partial nature, however. In general equilibrium, the positive supply of bonds will force the equity share of the average investor to lie inside the unit interval.

argued by Barro (1979) and many others. In Section 3 we return to further implications of this literature.

A near-zero elasticity of intertemporal substitution does not imply high risk aversion, however. Rational decision makers can be highly averse to planned variations in consumption over time and yet be highly tolerant of variations that result from stochastic movements in stock prices. On the other hand, we note that the annuity component in (4b) contains a risk adjustment. The permanent-income theory in its simplest form ignores uncertainty and recommends consumption of the entire expected return. When risk is considered, this would naturally be optimal only with risk neutrality, i.e. $\gamma = 0$. In general, the greater the risk aversion, the smaller the draw should be. More specifically, the safety buffer should correspond to half the expected return of the optimally chosen risky portion of the portfolio. For example, if the equity premium is 4 percent and the equity share has been optimally chosen as 60 percent, we can conclude that the annuity component of the optimal draw is 1.2 percentage points higher than the risk-free rate or, equivalently, 1.2 percentage points lower than the expected return on the entire portfolio. This difference is far from trivial.

Mean reversion in stock returns would make the correction smaller. Mean reversion has been noted by Fama and French (1988) and Poterba and Summers (1988) and discussed further in Campbell and Viceira (2002). However, Bansal and Yaron (2004) argue, convincingly, in our view, that a proper explanation of observed risk premia requires recognition of uncertainty, not only about current returns, but also about their long-term trend. Swanson (2016) implements Bansal and Yaron's specification in a complete macro model with apparent success and concludes similarly. Trend uncertainty would naturally add to the optimal risk correction of the annual draw. We thus do not believe that our derivation overstates it.

As we shall see in the following sections, the results in formulae (4a) and (4b) will have to be modified if taxes and public spending are smoothed or if the risk-free rate varies over time. However, the desirability of a safety buffer is a general finding, which we summarize as

Observation 1. *As a provision against risk, the annuity part of the optimal draw rate should be lower than the expected return on the portfolio.*

3. Backward smoothing

As the owner of an SWF, the government will want to use it to enhance government services and/or keep a lid on taxes. Barro (1979) and others have presented good arguments that both the tax system and the stream of government services ought to be smooth. This smoothness should work backward as well as forward. That is, policy makers should not only plan for smoothness in future services and tax rates, they should also avoid sudden changes from past patterns in response to unexpected shocks. In practice, policy makers often have no leeway when it comes to changing government services from one period to the next.

In our framework, forward smoothing is ensured by a low value of the elasticity of intertemporal substitution ε. Backward smoothing is provided to some extent by risk aversion because low risk taking limits the effects of negative random shocks. However, in the model as specified so far, the degree of (relative) risk aversion is independent of the level of consumption and wealth. A more natural assumption would be that this aversion becomes stronger the more strained the government's finances are compared to recent experience. This assumption can be approximated by introducing habit formation in the model. We naturally do not mean that policy decisions are governed by habits in a literal sense, but that models of habit formation offer a suitable technique for modelling variations in risk aversion and hence backward smoothing.[5]

The consumption literature distinguishes between external and internal habits. External habits refer to people's valuation of their own consumption relative to that of others: "catching up with the Jones'," cf. Abel (1990) or Campbell and Cochrane (1999). Internal habits refer instead to how people tend to get used to their standard of living and derive utility only from consumption over and above that standard. We believe this formulation of habit formation is the most relevant for our purpose because our decision maker is the government deciding for the entire nation, comparable to a representative agent.

[5] Please note that by the term "smoothing" we do not mean that the time series of consumption or draws from the fund becomes differentiable. We use it in the traditional macroeconomic way to imply a reduction in the volatility of the time series.

Constantindes (1990) has introduced habit formation in a model of portfolio investment in continuous time with expected-utility preferences. We extend his analysis to Epstein-Zin preferences by letting the consumption variable in the Bellman equation (2) be replaced by consumption y over and above a habit level x, so that $y = c - x$; and the habit level is assumed to start from an exogenously given initial level x_0 and to develop over time according to

(5) $$dx(t) = [bc(t) - ax(t)]dt \equiv by(t)dt + (b-a)x(t)dt, a \geq 0, b \geq 0, b < r + a.$$

Appendix A3 extends Constantinides' results for this specification to the case of Epstein-Zin non-expected utility. We show there that, in this case, the SWF can be thought of as consisting of two portfolios, one part with value $\mathcal{X}(t)$, providing safe financing for the minimum consumption level defined by the habit $x(t)$ and a remaining part with value $\mathcal{Y}(t)$ financing the rest. The portfolio with value $\mathcal{X}(t)$ needs to be risk free because otherwise a bad random draw could make consumption fall below $x(t)$, which would make utility drop to negative infinity. Safe funding of this minimum level of consumption turns out to require $\mathcal{X}(t) = x(t)/(r + a - b)$.

Relative risk aversion in this setup is defined by the transformed value function $U(W)$ defined after formula (2) above. Without habit formation, it is simply

$$RRA_{NH} = -\frac{U_{ww}W}{U_w} = \gamma.$$

For the analysis of behaviour under habit formation, this transformed value function is replaced by $U(\mathcal{Y})$. Thus, relative risk aversion is then defined as

(6) $$RRA_H = -\frac{U_{ww}W}{U_w} = -\left(\frac{U_{\mathcal{Y}\mathcal{Y}}\mathcal{Y}}{U_{\mathcal{Y}}}\right)\frac{W}{\mathcal{Y}} = \gamma\, W/\mathcal{Y} = \gamma\, W/(W - \mathcal{X}).$$

For $\mathcal{X} > 0$, this measure is unambiguously larger than for the case without habit formation. It is larger the smaller the relative difference between total wealth and the wealth needed to maintain safe funding for the minimum habit level of consumption. The more "squeezed" public finances are, the more risk averse policy makers will be. This is the property that we wanted our preference specification to have.

Under these conditions, the optimal equity share is no longer constant, and the optimal draw on the fund is no longer proportional to wealth. Instead, they follow the same rules that Constantinides shows for expected utility and which Appendix A3 shows hold also with the more general Epstein-Zin preferences. We summarise them as

Proposition 2. *With Epstein-Zin preferences and Constantinides habit formation as defined in (5), the optimal equity share and the optimal SWF draw are given by the following formulae:*

(7a) $$\alpha(t) = m\,[W(t) - \mathcal{X}(t)]/W(t)$$

and

(7b) $$c(t) = x(t) + \left(1 - \frac{b}{r+a}\right)\eta[W(t) - \mathcal{X}(t)],$$

where η is defined as in (4b).

Risk taking now clearly is limited by the need to be able to maintain the habit level $x(t)$ of consumption without risk. The equity share is proportional to the ratio of "free" wealth $W(t) - \mathcal{X}(t)$ to total wealth $W(t)$. An adverse development in the equity market should be followed by a reduction of the equity share. Put differently, the amount of wealth invested in the risky asset is a fixed proportion of the free wealth:

$$\alpha(t)W(t) = m[W(t) - \mathcal{X}(t)].$$

Draws from the fund must be large enough to permit consumption to at least equal the habit level of consumption. However, it also needs to be limited by the need to retain sufficient wealth to fund the habit level of consumption without risk. First, the "free" level of consumption (over and above the habit level) is a fixed proportion of only the "free" wealth. And second, this proportion is a little lower than the one in Section 2 because of the need to continuously set aside some money to ensure the continued safe funding of the habit level of consumption. This is the price to be paid for the opportunity to maintain at least the habit level of consumption no matter what happens to the return on risky assets.

For optimisation with habit formation to be feasible, the initial habit level obviously cannot be too large. As a minimum, it cannot exceed the riskless return on the entire fund. It also makes sense to assume $b \leq a$ because otherwise the habit level would tend to rise autonomously over time, which would have required an even larger riskless portfolio to finance habit consumption over time.

We summarize these insights as

Observation 2. *A wish to keep taxes and public services smooth over time should make risk aversion for SWF investment greater in general, and risk aversion should move in the opposite direction of the equity market. However, the extent of smoothing will have to be somewhat limited in order to be feasible.*

Cochrane (2017) argues that models with habit formation and models with non-expected utility (as well as other models used in macro-finance) in many ways provide similar modifications, technically speaking, of the classical power expected-utility model in terms of providing better explanations of observed data for equity premia and riskless rates. However, in the context of optimal portfolio choice and optimal spending decisions in a SWF context, our results in Proposition 2 demonstrate that these two modifications complement rather than substitute each other. Whereas the portfolio allocation is determined by the habit level, spending is determined by habits as well as the intertemporal elasticity of substitution provided by the non-expected utility model.

4. Rebalancing and long-term volatility

We furthermore note the following implication of (7a):

Observation 3. *If the government wants to maintain a smooth flow of taxes and government services, the rules for SWF portfolio rebalancing after asset price changes should be formulated so as to safeguard the funds needed to secure this smoothness.*

As a response to price changes in the risky part of the asset portfolio, the portfolio has to be rebalanced to obtain the optimal portfolio weights. Recall that, without habit formation, the risky share of the portfolio should always be the constant m. In this case, maintaining the optimal portfolio weights leads to counter-cyclical rebalancing: the fund buys more of the risky asset when its price falls and sells it when the price increases. With backward smoothing modelled as habit formation, counter-cyclical rebalancing may not always be optimal. Consider the following stylised example: An SWF has 100 to invest (for instance 100 billion dollars). It can be invested in a risky asset with price 100 or in a risk free asset also with price 100. The price of the risky asset falls to 98, making it necessary for the portfolio manager to rebalance the asset portfolio of the SWF. The portfolio holdings before and after rebalancing are illustrated in Table 1. In the first two cases, the price decrease makes the portfolio manager invest more in the risky asset (Δ risky investment > 0). While in the first case the fraction of the wealth invested in the risky asset is constant, in the second case the optimal fraction is slightly reduced after the price decrease. In the third case, with a lower risk aversion than in the second case, the price decrease lowers the total portfolio value so much that it starts to threaten the habit level of consumption. The investor responds to this threat by reducing the investment in the risky asset (Δ risky investment < 0). As in the second case, the optimal fraction of the wealth invested in the risky asset is reduced after the price fall. This example illustrates that, under habit formation, counter-cyclical rebalancing need not always be the optimal response to a price change in the risky asset.

Table 1. Illustration of portfolio rebalancing after a price decrease in the risky asset from 100 to 98. Parameter values are: r=0.05, μ=0.05, σ=0.2, a=0.3, and b=0.25.

			Non-Habit (γ=2)			
	Prisky	Prisk-free	Investment before rebalancing		Investment after rebalancing	
			Risky	Risk-free	Risky	Risk-free
Before price change	100	100	62.5	37.5		
After price change	98	100	61.25	37.5	61.72	37.03
Δ risky investment					0.47	

			Habit (γ=2, x=4)			
	Prisky	Prisk-free	Investment before rebalancing		Investment after rebalancing	
			Risky	Risk-free	Risky	Risk-free
Before price change	100	100	37.5	62.5		
After price change	98	100	36.75	62.5	37.03	62.22
Δ risky investment					0.28	

			Habit (γ=1.2, x=4)			
	Prisky	Prisk-free	Investment before rebalancing		Investment after rebalancing	
			Risky	Risk-free	Risky	Risk-free
Before price change	100	100	62.5	37.5		
After price change	98	100	61.25	37.5	61.2	37.55
Δ risky investment					-0.05	

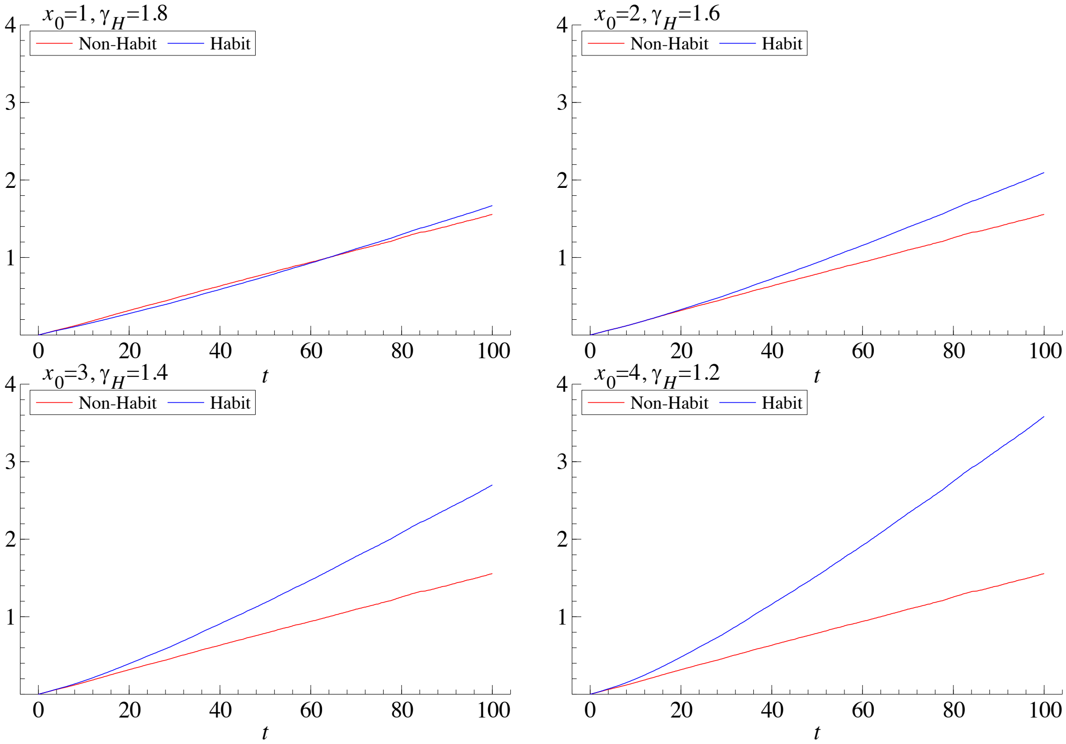

Figure 1. Variance of log-wealth at different time horizons t for the investor with non-expected utility, non-habit preferences (Non-habit) and for the investor with non-expected utility, habit preferences (Habit). Parameter values are $\mu = 0.05, \sigma = 0.20, r = 0.05, a = 0.30, b = 0.25, \rho = 0.03, \gamma = 2, \varepsilon = 0.1$, and $W_0 = 100$. The variances are estimated from 10,000 simulated observations at each point in time. We use 10 time points per year. The parameter x_0 shows the initial habit level and γ_H shows the γ coefficient for the investor with habit preferences.

As shown by Sundaresan (1989) and Constantinides (1990), smoothing, such as modelled in section 3 reduces the volatility of consumption on the short horizon. Indeed, habit formation has been invoked as a mechanism to help explain the empirical smoothness of consumption. The time-series of consumption has a lower volatility because the consumption level responds less to changes in wealth, compared to the standard model without habit formation. When the wealth increases, the consumption level increases relatively less and similarly for decreases in wealth. The direct effect of this smoothing is a higher variance on the fund value. Although this effect is partially countered by the more conservative investment portfolio, our numerical investigations indicate that the fund's value eventually becomes more volatile. This steeper rise in the long-horizon volatility of the fund value may then even be translated into a higher long-horizon volatility of consumption itself. Thus, although the short-horizon variance is lower with habit formation, the long-horizon variance can be higher for consumption as well as for the fund itself. These insights can be gleaned from inspection of the formulae involved. However, numerical simulations presented in Figure 1 show that the variance of future log-wealth for the case of habit-formation type of smoothing rises much more quickly with the length of the horizon than in the case without such smoothing (in the figure labelled "Non-Habit")[6]. The figure shows four different sets of parameter configurations. For all four configurations, the parameters are set so that the investor with habit preferences has the same initial portfolio composition as the investor without habit formation and a coefficient of relative risk aversion of $\gamma = 2$.

The long-horizon uncertainty of consumption is illustrated in Figure 2. Here, we plot the variances of log-consumption for different time horizons and for different parameter configurations. In three of the cases, we see lower consumption volatility for "short" horizons (say, less than 20 to 50 years) for the smoothing case than for the base case. However, for longer horizons the smoothing investor can face far more variation in consumption. This increase comes as a consequence of the riskier wealth illustrated in Figure 1.

This exercise teaches an important lesson: Smoothness carries a price. We can smooth current consumption by using the fund as a buffer. But then we tamper with the fund's principal. In so doing, we indirectly affect future consumption and hence future habit levels, which in turn influence consumption even further out. Short-term convenience carries long-term costs.

Provided the smoothing modelled as habits really is part of preferences, the trade-off between short-term smoothness and long-term uncertainty is done optimally. Figure 3 shows what would happen to the variance of log-consumption at various time horizons if the investment and spending decisions were separated so that the risky share of the portfolio were kept constant even though spending is based on the above implications of the habit model. The separation of spending and investment decisions leads to higher variability of consumption. At least as interesting is the fact that the separated rules eventually become inconsistent in all four examples, illustrated by the fact that the graph for the variance of log-consumption ends prematurely for the case of constant equity share (labelled "Habit mixed" in the figure). This phenomenon arises because keeping the risky share fixed fails to safeguard the funding of the minimal, habit-determined rates of future spending for some states of the world. Thus, the investor/consumer ends up in what Lax (2002) refers to as the insolvency range[7].

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[6] Calculations are performed using Ox, see Doornik (1999).

[7] Out of the 10,000 simulated consumption paths used to construct each of the four graphs for the variance of log-consumption (labeled "Habit mixed"), insolvency happens in 178, 227, 288, and 387 consumption paths, respectively, with the number increasing in the initial habit level x_o.

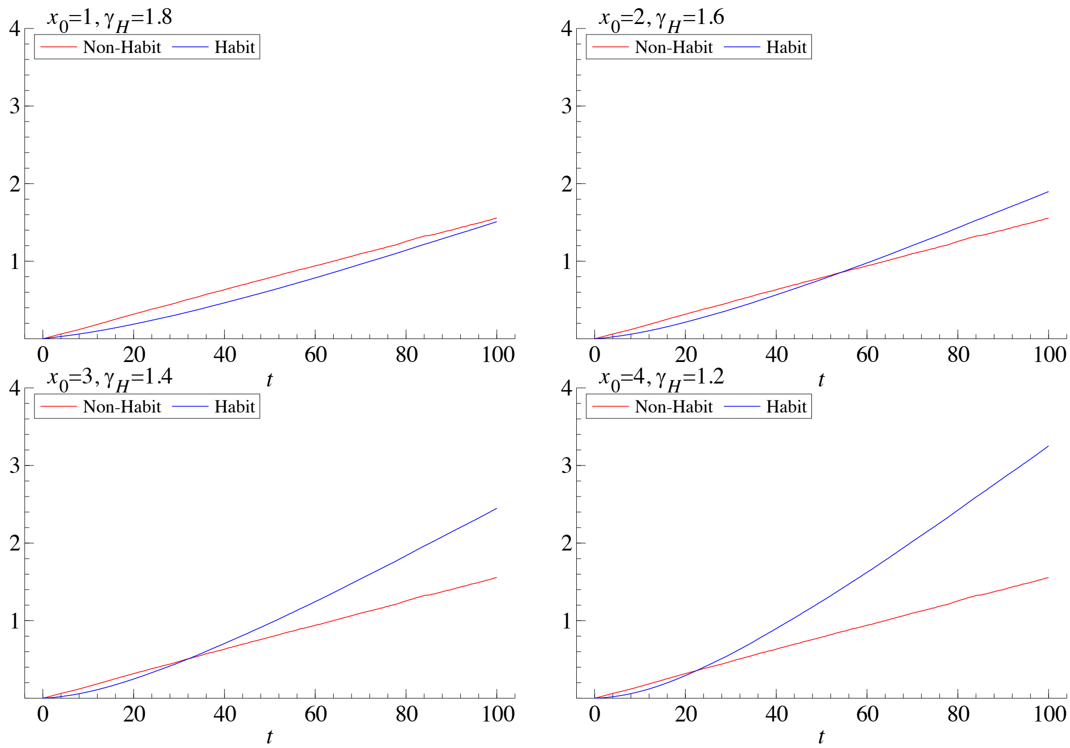

Figure 2. Variance of log-consumption at different time horizons t for the investor with non-expected utility, non-habit preferences (Non-habit) and for the investor with non-expected utility, habit preferences (Habit). Parameter values are $\mu = 0.05, \sigma = 0.20, r = 0.05, a = 0.30, b = 0.25, \rho = 0.03, \gamma = 2, \varepsilon = 0.1$, and $W_0 = 100$. The variances are estimated from 10,000 simulated observations at each point in time. We use 10 time points per year. The parameter x_0 shows the initial habit level and γ_H shows the γ coefficient for the investor with habit preferences.

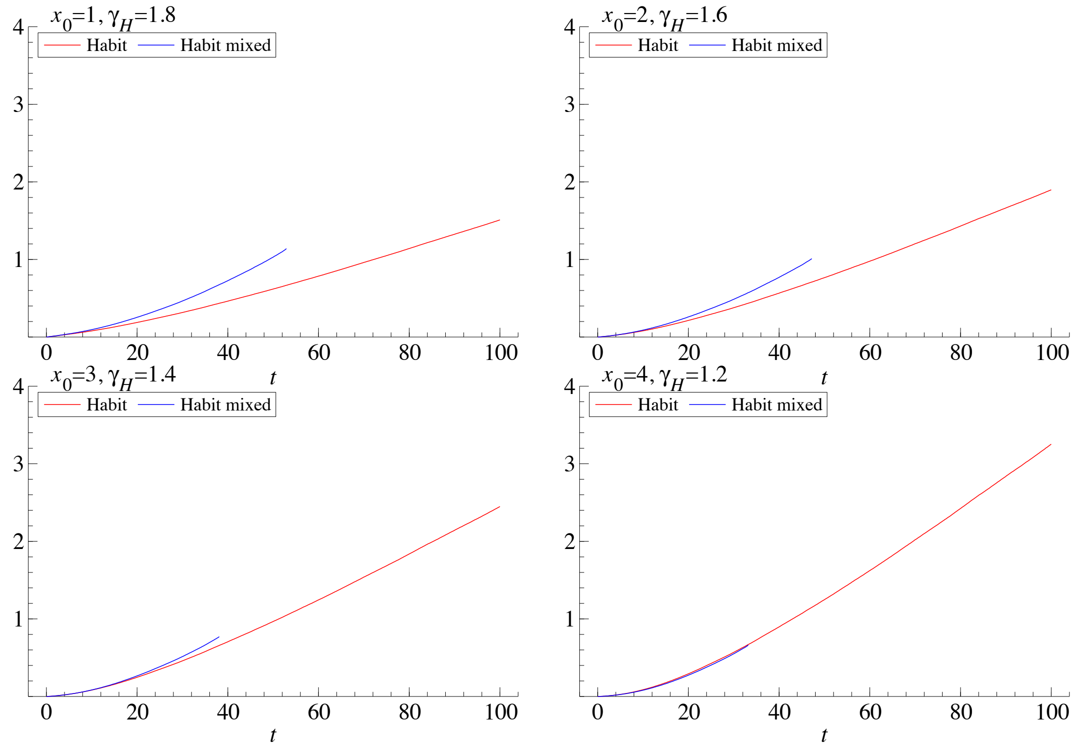

Figure 3. Variance of log-consumption at different time horizons t for two investors. The first is for the investor with non-expected utility, habit preferences (Habit). The second is for an investor with non-expected utility, habit preferences. This investor consumes as the habit-investor, but uses the portfolio weights for the investor with non-habit preferences (Mixed habit). Parameter values are $\mu = 0.05, \sigma = 0.20, r = 0.05, a = 0.30, b = 0.25, \rho = 0.03, \gamma = 2, \varepsilon = 0.1$, and $W_0 = 100$. The variances are estimated from 10,000 simulated observations at each point in time. We use 10 time points per year. The parameter x_0 shows the initial habit level and γ_H shows the γ coefficient for the investor with habit preferences.

We summarize these observations as

Observation 4. *The smoothing of consumption tends to carry a price in the form of a wider uncertainty of the long-run prospects for consumption. This uncertainty is mitigated by the optimal modification of portfolio rebalancing, but it is not removed.*

5. Time-varying risk-free rates

In this section, we return to the setting in section 2, but now extend the analysis to account for interest-rate uncertainty. In practice, risk-free rates are not constant over time. In fact, recent decades have seen a long global trend of falling real interest rates since the mid-1980s, as documented, e.g. by King and Low (2014) and Rachel and Smith (2015). With a less than unit elasticity of intertemporal substitution, a permanent drop in the risk-free rate should obviously imply a corresponding reduction of the optimal draw on an SWF, as indicated by (4b). However, some authors, such as the OECD (2014), argue that the recent long decline is likely to be reversed, at least partially. We now turn to the question of what the optimal draw should be under these conditions.

For this purpose, we need a specification of the dynamically stochastic variation in the instantaneous risk-free rate. For tractability, we use the mean-reverting diffusion process introduced by Vasicek (1977):

(14) $$dr(t) = \theta\big(r^* - r(t)\big)dt + \zeta d\xi(t), \theta \geq 0,$$

where $\xi(t)$ is a Wiener process, possibly correlated with $B(t)$, r^* is the steady-state riskless return rate, θ the speed of return towards this rate after deviations (also known as the force of gravity), and ζ is the instantaneous standard deviation of *r(t).* This specification is consistent with the OECD's expectation.

When the risk-free rate varies over time, the question also arises whether the equity premium stays constant or whether the expected equity return stays constant instead, so that the equity premium varies inversely with the risk-free rate. We choose to remain agnostic about this issue and use a specification of the equity premium that encompasses both cases:

(15) $$\mu(t) = \mu\big(r(t)\big) = \lambda\bar{\mu} + (1-\lambda)\big(\varphi - r(t)\big),\ 0 \leq \lambda \leq 1,$$

where $\bar{\mu}$ denotes the risk premium if constant ($\lambda = 1$) and φ the equity return if that is constant instead ($\lambda = 0$).

With this modification, the diffusion process for wealth now becomes

(16) $$dW(t) = \{[r(t) + \alpha(t)\mu(t)]W(t) - c(t)\}dt + \alpha(t)\sigma W(t)dB(t).$$

We finally note that the Bellman equation (2) also needs to be modified to allow for the time variation in the riskless rate:

(17) $$0 = \max_{c(t),\alpha(t)} \left\{c(t)^{1-\delta} - \rho U(W(t), r(t))^{(1-\delta)/(1-\gamma)} + \frac{1}{dt}[E_t dU(W(t), r(t))]^{(1-\delta)/(1-\gamma)}\right\}.$$

As shown in Appendix A1, the time variability of the risk-free rate does not change the result that the optimal draw is proportional to the wealth level. However, the proportionality factor is now a function of the risk-free rate. This function is the solution to a non-linear, second-order differential equation. Although a closed-form solution to this equation may exist, our interest focuses on its semi-elasticity with respect to the risk-free rate. As shown in Appendix A4, this elasticity can be approximated around the steady-state risk-free rate of return r^* by means of the method used by Campbell and Viceira (2002) in their Chapter 5. Appendices A1 and A4 thus prove the following

Proposition 5. *If the risk-free return varies over time according to (14), the optimal SWF draw, as a ratio of wealth, should respond to time variations in the risk-free rate according to*

(18) $$d\ln\eta(r)/dr \cong (1-\varepsilon)\,[1-(1-\lambda)\alpha^*]/(\eta^* + \theta) < (1-\varepsilon)/\eta^* = d\ln\eta^*/dr^*,$$

where η^* denotes the value of η if the risk-free rate is constant at its steady-state value, as defined in (4b), and α^* is the optimal equity share for $r = r^*$.

This intuitive formula shows that the reaction to dynamically stochastic variation in the risk-free rate should have the same sign as the comparative-static response, but be smaller in absolute value because of the expectation of an eventual mean reversion[8]. The difference between the two responses depends positively on λ and negatively on θ. $\lambda = 0$ means that an increase in the risk-free rate lowers the equity premium and thus reduces the incentive for risk taking, so that the portfolio return is raised by somewhat less than the rise in the riskless rate, which in turn dampens the effect on the optimal draw rate. In contrast, $\lambda = 1$ means that the equity premium is unaffected by the variation in the risk-free rate, so that this dampening disappears.

The effect on the optimal draw rate is furthermore dampened by θ, the speed of the mean-reversion adjustment. Thus, the expectation that the risk-free rate eventually returns to normal allows for a smoothing of SWF draws relative to movements in this rate. This is true even if the return path is rocky in the sense of a large ζ. The larger the speed of this mean-reversion adjustment, the stronger the smoothing. If the deviation from the steady state is truly ephemeral ($\theta \to \infty$), the smoothing is complete. If, however, the mean reversion is expected to take very long, the smoothing should be very modest.

We summarize this insight as

Observation 5. *The rule for SWF draws should smooth over changes in the risk-free return provided they are temporary. The strength of the optimal smoothing depends on the expected speed of return to normal, even if the return path is rocky. The smoothing should be even stronger if the movements in the risk-free return leaves the expected return on equity unchanged.*

Proposition 6. *If the risk-free return varies over time according to (14), the optimal equity share becomes*

(19) $$\alpha\big(r(t)\big) = \frac{\mu(r(t))}{\gamma\sigma^2} - (1 - 1/\gamma)\beta\,[1 - (1 - \lambda)\alpha^*]/(\eta^* + \theta),$$

[8] We also note that, in both cases, the response has the sign as $1 - \varepsilon$. This is because the effect on consumption is dominated by the income effect when $\varepsilon < 1$. When $\varepsilon > 1$, the substitution effect dominates, so that the investor will want to postpone consumption when the risk-free rate rises, so that current consumption falls. We maintain, however, that SWF decision makers tend to think mainly in terms of the income effect in this regard, which is consistent with our presumption that epsilon can be assumed to be quite small in the context of SWF management.

where β denotes the theoretical regression coefficient of $\zeta d\xi$ on σdB.

Again, the proof is contained in Appendices A1 and A4. The first term indicates an inverse relationship between the risk-free rate and the equity share if a drop in the risk-free rate implies a higher equity premium. That would be the case if, as argued by Hall (2016) and Caballero, Farhi, and Gourinchas (2008) regarding the recent long decline in real interest rates, such a decline is due to an increase in the risk aversion of the marginal investor in global markets.

Compared to (4a), this formula furthermore has an additional term which reflects the dynamic risk of future changes in the risk-free rate. We assume $\beta \leq 0$ because a drop in the risk-free rate tends to coincide with higher stock valuations, other things equal. The entire second term, including the minus sign in front, is positive provided $\gamma > 1$, which we view as the normal case. Thus, the risk of a future drop in the risk-free rate calls for a somewhat higher equity share as a dynamic hedge against such a drop. Not surprisingly, this effect is weaker if risk aversion is lower and is actually reversed if risk aversion is very low ($\gamma < 1$)[9].

We summarize these insights as

Observation 6. *A temporary change in the risk-free rate calls for an increase in the equity share if this change leaves the expected equity return unchanged. For a reasonably risk-averse investor, the uncertainty of future risk-free rates furthermore calls for a somewhat higher equity share as a dynamic hedge provided a drop in the risk-free rate is associated with a higher equity return.*

6. Conclusions and future work

The classical theory of portfolio management provides helpful insights into the issues involved in the management of SWFs in advanced-economy countries. We have highlighted some of these insights and extended them in directions that we consider important.

[9] Indeed, if the inverse link between innovations in equity prices and the risk-free rate is strong enough, a risk neutral investor could decide to hold a short net equity position so as to take advantage of the ensuing fall in equity prices as well as the rise in the riskless rate should the latter occur. We do not pursue this extreme case, however.

Our extension starts with the introduction of Epstein-Zin non-expected utility preferences in continuous time, which offers an explanation of how policy makers that are anxious to preserve the fund for future generations nevertheless can be willing to take on considerable financial risk. However, risk taking calls for caution in drawing from the fund in the form of a risk correction relative to the expected portfolio return.

We also study backward as well as forward smoothing of the draws on the fund as a way to accommodate the generally desired smoothness of tax rates and public services. For this purpose, we use habit formation as a technical tool, which implies that risk aversion becomes higher when public finances are strained and vice versa. Not surprisingly, such smoothing calls for less risk taking and that risk aversion varies inversely with equity returns. Moreover, the optimal equity share may vary positively with equity returns in such a way that a drop in equity prices in some cases may call for selling rather than buying of equities. Backward smoothing tends to *raise* the long-horizon variance of the fund value and can even raise the long-horizon variance of the annual draws.

If the riskless rate varies over time, the annual draws on the fund should vary in the same direction; however, it may be optimal to smooth the time variation in the draws relative to those of the risk-free rate if the latter can be expected to eventually return to normal.

Although we believe our analysis should be interesting and important for advanced-economy SWF management, we realize that it only scratches the surface. Just as most households receive labour income in addition to capital income, a government collects taxes. Furthermore, tax revenues as well as spending needs tend to vary with the business cycle, which in turn tends to correlate with stock returns as well as real interest rates. Active, countercyclical fiscal policy adds further volatility to these fluctuations. In the context of an SWF, this means that the policy makers not only want to smooth the draws on the fund relative to returns; they may even want the draws to fluctuate in the opposite direction. Such opposite fluctuations would translate into attacks on the fund's principal in bad economic times, which could add to the long-term fluctuations and jeopardize the fund's solvency if caution is not taken.

We intend to take up these and other issues in our future research. As is well known from the context of household behaviour, the presence of non-capital income sources presents no new complication if markets are complete. However, this assumption is especially problematic in the context of government finances, as moral hazard, dynamic inconsistency, and poor contract enforceability mean that a government's future tax revenues cannot be capitalized. As is well known, exact analytical solutions are then no longer available except in uninteresting cases like quadratic utility.

Viceira (2001) has worked out approximate solutions for optimal consumption and investment for an individual with stochastic labour income under the assumption that permanent changes in labour income are the main determinants of spending behaviour. Benzoni, Collin-Dufresne, and Goldstein (2007) similarly study co-integration between stock and labour markets. However, these approaches are unsuitable for studying the implications of cyclical variations in government spending and revenues. We thus expect that we will need to rely on simulation exercises to obtain numerical solutions.

References

Abel, A. B. (1990). "Asset prices under habit formation and catching up with the Joneses." *American Economic Review 80* (1), 38–42.

Alhashel, B. (2015). "Sovereign Wealth Funds: A literature review." *Journal of Economics and Business 78, 1 – 13.*

Bansal, R. and A. Yaron (2004). "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance 59* (4), 1481 – 1509.

Barro, R. J. (1979). "On the determination of the public debt." *Journal of Political Economy 87*, 940–971.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007). "Portfolio choice over the life-cycle when the stock and labor markets are cointegrated." *Journal of Finance 62* (5), 2123–2167.

Bernstein, S., J. Lerner, and A. Schoar (2013). "The Investment Strategies of Sovereign Wealth Funds." *Journal of Economic Perspectives, 27* (2), 219 – 237.

Caballero, R.J., E. Farhi, and P.-O. Gourinchas (2008). "An Equilibrium Model of 'Global Imbalances' and Low Interest Rates." *American Economic Review 98* (1), 358 – 93.

Campbell, J. and A. Deaton (1989). "Why is consumption so smooth?" *Review of Economic Studies 56*, 357–373.

Campbell, J. and J. Cochrane (1999). "By force of habit: A consumption-based explanation of aggregate stock market behavior." *Journal of Political Economy 107* (2), 205–251.

Campbell, J. Y. and L. Viceira (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors* (first ed.). Oxford University Press.

Carroll, C. and O. Jeanne (2009). “A Tractable Model of Precautionary Reserves, Net Foreign Assets, or Sovereign Wealth Funds.” NBER Working Paper no. 2824.

Carroll, C., J. Overland, and D. Weil (2000). “Saving and growth with habit formation.” *American Economic Review 90*, 341–355.

Choudhry, M. (2007). *Bank Asset and Liability Management: Strategy, Trading, Analysis* (first ed.). Wiley Finance.

Cochrane, J.H. (2017). “Macro-Finance.” *Review of Finance 21 (3)*, 945 – 985.

Collier, P. and J.W. Gunning (2005). “Asset Policies During an Oil Windfall: Some Simple Analytics.” *The World Economy 28* (10), 1401 – 1415.

Constantinides, G. (1990). “Habit formation: A resolution of the equity premium puzzle.” *Journal of Political Economy 98*, 519–543.

Corden WM (1984). "Boom Sector and Dutch Disease Economics: Survey and Consolidation". *Oxford Economic Papers*. **36**: 362.

Doornik, J. (1999). *Object-Oriented Matrix Programming Using Ox.* London: Timberlake Consultants Press and Oxford (www.nuff.ox.ac.uk/Users/Doornik).

Dreassi, A., S. Miani, and A. Paltrinieri (2017). “Sovereign Pension and Social Security Reserve Funds: A Portfolio Analysis”, *Global Financial Journal*, in press.

Epstein, L. G. and S.E. Zin (1991). “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis.” *Journal of Political Economy 99* (2), 263 – 286.

Fama, E. F. and K. R. French (1988). “Permanent and temporary components of stock prices.” *Journal of Political Economy 96*, 246–273.

Flavin, M. (1985). “The excessive sensitivity of consumption to current income - liquidity constraints or myopia?” *Canadian Journal of Economics 1* (February), 117–136.

Giovannini, A. and P. Weil (1989). “Risk aversion and intertemporal substitution in the capital asset pricing model.” NBER Working Paper no. 2824.

Greenspan, Alan (2007): *The Age of Turbulence: Adventures in a New World.* The Penguin Press.

Guerra-Salas, J.F. (2014). “Government Investment and the Business Cycle in Oil-Exporting Countries.” Fordham University.

Hall, R. E. (1978). "Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence." *Journal of Political Economy 86*, 971–987.

Hall, R. E. (1988). "Intertemporal Substitution in Consumption." *Journal of Political Economy 96* 339 – 357.

Hall, R. E. (2016). "The Role of the Growth of Risk-Averse Wealth in the Decline of the Safe Real Interest Rate." Stanford University.

Johan, S.A., A. Knill, and N. Mauck (2013). "Determinants of Sovereign Wealth Fund Investment in Private Equity vs Public Equity," *Journal of International Business Studies, 44* (2), 155 – 172.

King, M. and D. Low (2014). "Measuring the 'world' real interest rate." NBER Working Paper 19887.

Lax, Y. (2002). "Habit formation and lifetime portfolio selection." Goldman Sachs Asset Management.

Merton, R. C. (1969). "Lifetime portfolio selection under uncertainty: The continuous time case." *Review of Economics and Statistics 51*, 247–257.

Norwegian Ministry of Finance (2015). *Finanspolitikk i en oljeøkonomi (Fiscal Policy in an Oil Economy)*, Official Norwegian Report NOU 2015:9.

Norwegian Ministry of Finance (2016). *Aksjeandelen i Statens pensjonsfond utland (The Equity Share of the Government Pension Fund Global)*, Official Norwegian Report NOU 2016:20.

OECD (2014). "The Long-Term Global Outlook for Interest Rates." Report commissioned by the Norwegian Ministry of Finance.

Phelps, E. S. (1962). "The accumulation of risky capital: A sequential utility analysis." *Econometrica 30*, 729–743.

Poterba, J. and L. Summers (1988). "Mean reversion in stock returns: Evidence and implications." *Journal of Financial Economics 22*, 27–60.

Rachel, L. and T. Smith (2015). "Secular drivers of the global real interest rate." IMF Staff Working Paper No. 571.

Sá, F. and F. Viani (2013). "Shifts in Portfolio Preferences of International Investors: An Application to Sovereign Wealth Funds." *Review of International Economics 21* (5), 868–885.

Samuelson, P. A. (1969). “Lifetime portfolio selection by dynamic stochastic programming.” *Review of Economics and Statistics 51* (3), 239–246.

Sundaresan, S. (1989). “Intertemporally dependent preferences and the volatility of consumption and wealth.” *Review of Financial Studies 2*, 79–89.

Svensson, L.E.O. (1989). “Portfolio Choice with Non-Expected Utility in Continuous Time.” *Economic Letters 30*, 313 – 317.

Swanson, E.T. (2016). “A Macroeconomic Model of Equities and Real, Nominal, and Defaultable Debt.” University of California, Irvine.

Thimme, J. (2016). “Intertemporal Substitution in Consumption: A Literature Review.” *Journal of Economic Surveys 31* (1), 226 – 257.

van den Bremer, T., van der Ploeg, F. and S. Wills (2016). “The Elephant In The Ground: Managing Oil and Sovereign Wealth.” *European Economic Review, 82* (2016), 113 – 131.

van der Ploeg, F. and A.J. Venables (2011). “Harnessing Windfall Revenues: Optimal Policies for Resource-Rich Developing Economies.” *The Economic Journal, 121* (551), 1 – 30.

Van Wijnbergen, Sweder (1984). "The 'Dutch Disease': A Disease After All?". *The Economic Journal*. 94 (373): 41 – 55.

Vasicek, O. A. (1977). “An equilibrium characterization of the term structure.” *Journal of Financial Economics 5* (2), 177–188.

Viceira, L. M. (2001). “Optimal portfolio choice for long-horizon investors with nontradable labor income.” *Journal of Finance 56* (2), 433–470.

Yang, W. (2015). “Intertemporal Substitution and Equity Premium.” *Review of Finance 20* (1), 403–445.

Appendix: Optimal investment and consumption with Epstein-Zin preferences in continuous time, with time-changing risk-free rate or habit formation

A1. General setup

We start with the value function for Epstein-Zin preferences in discrete time:

(A1.1) $$e^{\rho t}V_t(W_t,r_t)^{1-\delta}=\max_{c_t,\alpha_t}\left\{e^{\rho t}c_t^{1-\delta}\Delta t+e^{\rho(t+\Delta t)}E_t[V_{t+\Delta t}(W_{t+\Delta t},r_{t+\Delta t})^{1-\gamma}]^{(1-\delta)/(1-\gamma)}\right\}.$$

Define $U_t(W_t,r_t)=V_t(W_t,r_t)^{1-\gamma}$. Then, the value function may be rearranged and written as

$$0=\max_{c_t,\alpha_t}\left\{e^{\rho t}c_t^{1-\delta}\Delta t+e^{\rho(t+\Delta t)}E_t[U_{t+\Delta t}(W_{t+\Delta t},r_{t+\Delta t})]^{(1-\delta)/(1-\gamma)}\right.$$
$$\left.-e^{\rho t}U_t(W_t,r_t)^{(1-\delta)/(1-\gamma)}\right\}.$$

Division by $e^{\rho t}\Delta t$ and taking the limit as $\Delta t\searrow 0$ now gives us the Bellman equation in continuous time:

(A1.2) $$0=\max_{c,\alpha}\left\{c^{1-\delta}-\rho U(W,r)^{(1-\delta)/(1-\gamma)}+\frac{1}{dt}[E_t dU(W,r)]^{(1-\delta)/(1-\gamma)}\right\}.$$

where dates t have been omitted except for the expectations operator.

The dynamic budget constraints are as follows:

(A1.3) $$dW=[(r+\alpha\mu)W-(1+\tau)c]dt+\alpha W\sigma dB(t),\ \mu=\lambda\bar{\mu}+(1-\lambda)(\varphi-r),$$

(A1.4) $$dr=\theta(r^*-r)dt+\zeta d\xi(t),\theta\geq 0.$$

Here, B and ξ are Wiener processes for equity returns and the risk-free rate, respectively. μ is the equity premium, which is time invariant and equal to $\bar{\mu}$ if $\lambda=1$. If $\lambda=0$, the expected equity return, is time invariant instead and equal to φ, so that the equity return moves one for one in opposite directions of the risk-free rate. τ is a consumption tax whose significance will be apparent in section A3.

We follow Campbell and Viceira (2002) by working with a second-order expansion of the value function under the conjecture that

(A1.5) $$V(W,r) = \Phi(r)W.$$

Then, ignoring higher-order terms, we have

$$\begin{aligned} dV &= V_W dW + V_r dr + \frac{1}{2}V_{WW}dW^2 + \frac{1}{2}V_{rr}dr^2 + V_{Wr}dWdr \\ &= \Phi(r)dW + \Phi'(r)Wdr + \tfrac{1}{2}\Phi''(r)Wdr^2 + \Phi'(r)dWdr. \end{aligned}$$

Substituting from the budget constraints, this can be further written as

$$\begin{aligned} dV = \Phi(r)W\Bigg\{\bigg[&\big(1-\alpha(1-\lambda)\big)r + \alpha(\lambda\bar{\mu} + (1-\lambda)\varphi) - (1+\tau)\,c/W + \frac{\Phi'}{\Phi}\theta(r^* - r) + \frac{1}{2}\frac{\Phi''}{\Phi}\zeta^2 \\ &+ \frac{\Phi'}{\Phi}\alpha\sigma\zeta\rho_{B\xi}\bigg]dt + \alpha\sigma dB + \frac{\Phi'}{\Phi}\zeta d\xi\Bigg\}, \end{aligned}$$

where $\rho_{B\xi}$ is the correlation coefficient between dB and $d\xi$.

By Itô's lemma, this gives us

$$\begin{aligned} E_t dU = (1-\gamma)U\Bigg\{&\big(1-\alpha(1-\lambda)\big)r + \alpha(\lambda\bar{\mu} + (1-\lambda)\varphi) - (1+\tau)\,c/W + \frac{\Phi'}{\Phi}\theta(r^* - r) + \frac{1}{2}\frac{\Phi''}{\Phi}\zeta^2 \\ &+ \frac{\Phi'}{\Phi}\alpha\sigma\zeta\rho_{B\xi} - \frac{1}{2}\gamma\left[\alpha^2\sigma^2 + \left(\frac{\Phi'}{\Phi}\right)^2\zeta^2 + 2\frac{\Phi'}{\Phi}\alpha\sigma\zeta\rho_{B\xi}\right]\Bigg\}dt. \end{aligned}$$

Using polynomial expansion, ignoring higher-order terms of dt, this furthermore gives us

$$\begin{aligned} E_t dU^{(1-\delta)/(1-\gamma)} = (1-\delta)U\Bigg\{&\big(1-\alpha(1-\lambda)\big)r + \alpha(\lambda\bar{\mu} + (1-\lambda)\varphi) - (1+\tau)\,c/W + \frac{\Phi'}{\Phi}\theta(r^* - r) \\ &+ \frac{1}{2}\frac{\Phi''}{\Phi}\sigma^2 + \frac{\Phi'}{\Phi}\alpha\sigma\zeta\rho_{B\xi} - \frac{1}{2}\gamma\left[\alpha^2\sigma^2 + \left(\frac{\Phi'}{\Phi}\right)^2\zeta^2 + 2\frac{\Phi'}{\Phi}\alpha\sigma\zeta\rho_{B\xi}\right]\Bigg\}dt. \end{aligned}$$

After dividing through by $U^{(1-\delta)/(1-\gamma)}$, this allows us to write the Bellman equation as

(A1.6) $$0 = \max_{c,\alpha}\{c^{1-\delta}\Phi^{-(1-\delta)}W^{-(1-\delta)} - \rho$$
$$+(1-\delta)\left[\big(1-\alpha(1-\lambda)\big)r + \alpha(\lambda\bar{\mu} + (1-\lambda)\varphi) - \frac{1}{2}\gamma\alpha^2\sigma^2 - (1+\tau)\,c/W\right]$$
$$+(1-\delta)\frac{\Phi'}{\Phi}[\theta(r^* - r) + \alpha(1-\gamma)\beta\sigma^2] + \frac{1}{2}(1-\delta)\left[\frac{\Phi''}{\Phi} - \gamma\left(\frac{\Phi'}{\Phi}\right)^2\right]\zeta^2\Bigg\},$$

where $\beta \equiv \rho_{B\xi}\zeta/\sigma$ is the theoretical regression coefficient of $d\xi$ on dB.

The first- order condition for c is now easily obtained as

(A1.7) $$c = (1+\tau)^{-1/\delta}\Phi(r)^{1-1/\delta}W.$$

Define $\eta(r) = (1+\tau)^{1-1/\delta}\Phi(r)^{1-1/\delta}$, so that

(A1.8) $$c = \left(\frac{1}{1+\tau}\right)\eta(r)W.$$

Then, obviously,

$$\Phi(r) = (1+\tau)\eta(r)^{-\delta/(1-\delta)},$$

which implies

(A1.9a) $$\frac{\Phi\prime}{\Phi} = -\frac{\delta}{1-\delta}\eta'/\eta$$

(A1.9b) $$\frac{\Phi\prime\prime}{\Phi} = -\frac{\delta}{1-\delta}\left[\left(\frac{1}{1-\delta}\right)(\eta'/\eta)^2 - \eta''/\eta\right].$$

Then,

$$\frac{1}{2}(1-\delta)\left[\frac{\Phi''}{\Phi} - \gamma\left(\frac{\Phi'}{\Phi}\right)^2\right]\zeta^2 = \frac{1}{2}\delta\left[\left(\frac{1-\gamma\delta}{1-\delta}\right)(\eta'/\eta)^2 - \eta''/\eta\right]\zeta^2.$$

Substituting, we can now write the Bellman equation as

$$0 = \max_{\alpha}\Big\{\delta\eta - \rho + (1-\delta)\left[\big(1-\alpha(1-\lambda)\big)r + \alpha(\lambda\bar{\mu} + (1-\lambda)\varphi) - \tfrac{1}{2}\gamma\alpha^2\sigma^2\right] - \delta[\theta(r^* - r) +$$
$$\alpha(1-\gamma)\beta\sigma^2]\,\eta'/\eta + \tfrac{1}{2}\delta\left[\left(\tfrac{1-\gamma\delta}{1-\delta}\right)(\eta'/\eta)^2 - \eta''/\eta\right]\zeta^2\Big\}.$$

Maximization with respect to α gives

(A1.10) $$\alpha = \frac{\mu(r)}{\gamma\sigma^2} - \left(\frac{\delta}{1-\delta}\right)\left(\frac{1-\gamma}{\gamma}\right)\beta\,\eta'/\eta\,,\;\; \mu(r) = \lambda\bar{\mu} + (1-\lambda)(\varphi - r).$$

Define now $\mu^* = \mu(r^*)$, so that $\mu(r) = \mu^* + (1-\lambda)(r^* - r)$.

After substitution of these formulae, the Bellman equation boils down to

$$0 = \eta - \varepsilon\rho - (1-\varepsilon)r - \theta(r^* - r)\,\eta'/\eta$$

$$-(1-\varepsilon)\left[\frac{\mu^*}{\gamma\sigma^2} + \frac{1-\lambda}{\gamma\sigma^2}(r^* - r) - \frac{1-1/\gamma}{1-\varepsilon}\beta\,\eta'/\eta\right][\mu^* + (1-\lambda)(r^* - r)]$$

(A.1.11) $$+(1-\varepsilon)\frac{1}{2}\left[\frac{\mu^*}{\gamma\sigma^2} + \frac{1-\lambda}{\gamma\sigma^2}(r^* - r) - \frac{1-1/\gamma}{1-\varepsilon}\beta\,\eta'/\eta\right]^2\gamma\sigma^2$$

$$-\{1-\gamma)\left[\frac{\mu^*}{\gamma\sigma^2} + \frac{1-\lambda}{\gamma\sigma^2}(r^* - r) - \frac{1-1/\gamma}{1-\varepsilon}\beta\,\eta'/\eta\right]\beta\sigma^2\,\eta'/\eta$$

$$+\frac{1}{2}\left[\left(\frac{\gamma-\varepsilon}{1-\varepsilon}\right)(\eta'/\eta)^2 - \eta''/\eta\right]\zeta^2,$$

where $\varepsilon \equiv 1/\delta$ is the elasticity of intertemporal substitution.

A2. Constant risk-free rate

In this case, $r = r^*$ og $\zeta^2 = 0$. Thus, Φ and η are constants. From (A1.10) and (A1.11), we then find

(A2.1) $$\alpha = \frac{\mu}{\gamma\sigma^2} \equiv m$$

(A2.2) $$\eta = \varepsilon\rho + (1-\varepsilon)\left(r + m\mu - \frac{1}{2}\gamma m^2\sigma^2\right) = \varepsilon\rho + (1-\varepsilon)\left(r + \frac{1}{2}m\mu\right).$$

From (A1.8), we see that the optimal consumption level as a share of wealth is η divided by one plus the consumption tax rate, if any. These results generalize the ones of Merton (1971). To our knowledge, they were first derived by Svensson (1989), with the exception of the consumption tax. However, we find our slightly different derivation method to be better amenable to applications with a time-varying riskless rate and habit formation.

A3. Habit formation

In this section, we assume a constant risk-free rate. We model habit formation by letting $y \equiv c - x$ replace c in the preference ordering, where x is the habit level. Following Constantinides (1990), we assume that the habit level moves over time according to the constraint

(A3.1) $$x = e^{-at}x_0 + b\int_0^t e^{a(s-t)}c(s)ds\,, a \geq 0, b \geq 0, b < r + a,$$

where x_0 is exogenous, which implies

(A3.2) $$dx = (bc - ax)dt \equiv bydt + (b-a)xdt.$$

Constantinides has derived formulas for optimal consumption and equity with this specification of habit formation with expected utility. We now proceed to show that these results also hold under the specification of non-expected utility in A1.

Our proof starts with the observation that, with our specification of habit formation, utility becomes infinitely negative if $y = c - x < 0$. With normally distributed equity returns, there is a non-zero probability that the return on equity at some point will be less than any arbitrarily negative number. It then follows that the habit level of consumption must be funded by investment in the safe asset only.

Thus, we may think of the investor-consumer as managing two separate portfolios. One portfolio, whose value we denote $\mathcal{X}$, is held for the exclusive purpose of funding the habit level x. The other portfolio, denoted $\mathcal{Y}$, is invested in order to fund the "surplus" level of consumption, y. We continue to let W denote total wealth, so that

(A3.3) $$W = \mathcal{X} + \mathcal{Y}.$$

Because the habit level in general will move over time, the value of its funding portfolio, $\mathcal{X}$, will generally need to move as well. Thus, it is not sufficient to maintain $\mathcal{X} = x/r$ because then the entire return on this portfolio will be absorbed by the current habit level. This means that the investor-consumer must make a regular transfer from the "surplus" portfolio $\mathcal{Y}$ to the "habit" portfolio $\mathcal{X}$. If $a > b$ these transfers may be negative, but that does not change the argument.

Let v denote the transfer. We may then specify the budget constraints for the two portfolios as

(A3.4) $$d\mathcal{Y} = [(r + \alpha'\mu)\mathcal{Y} - y]dt - vdt + \alpha'\mathcal{Y}\sigma dB$$

and

(A3.5) $$d\mathcal{X} = r\mathcal{X}dt - xdt + \ vdt.$$

where α' now denotes the equity share of the surplus portfolio $\mathcal{Y}$, not necessarily constant. Here, we have implicitly assumed that the consumption tax on c is zero.

Based on Constantinides' results for the case of expected utility, we conjecture that optimization requires

(A3.6) $$\mathcal{X} = x/(r + a - b).$$

Under this conjecture, (A3.5) implies

$$dx = (b - a)xdt + (r + a - b)vdt.$$

This condition is equivalent to the flow condition for habit formation (A3.2) if and only if

(A3.7) $$v = \left(\frac{b}{r+a-b}\right)y.$$

Consider now an arbitrary path for y. Given (A3.7) and (A3.4), this path for v is just sufficient to fund the path of the habit level as given by (A3.2). If, at any time, a lower value of v is chosen, the implied size of the habit portfolio $\mathcal{X}$ will be insufficient to fund the habit level implied by (A3.2), so it is not feasible. If, on the other hand, a higher value is chosen, it will be wasted because the habit portfolio will become excessively large. Thus, (A3.7) indicates the optimal path for the transfer amount v for an arbitrary path of y.

Next, consider the optimal surplus consumption path y. Note, then, that the optimal transfer level v, given by (A3.7), is linear in y. Thus, it has the same form as a consumption tax with

$$\tau = b/(r+a-b).$$

From (A1.8), we then see that the optimal level of surplus consumption must be

$$y = \left(\frac{r+a-b}{r+a}\right)\eta\mathcal{Y}.$$

Thus, using (A3.3), the definition of y, and the conjecture in (A3.6), we find

(A3.8) $$c = x + \left(\frac{r+a-b}{r+a}\right)\eta[W - x/(r+a-b)].$$

For the conjecture in (A3.6) to hold, it suffices that the investor-consumer at time 0 sets aside

(A3.9) $$\mathcal{X}_0 = x_0/(r+a-b)$$

as the initial habit portfolio.

Thus, habit formation according to (A3.1) requires two kinds of set-aside into the habit portfolio. First, at time 0, the investor-consumer must set aside the initial amount in

(A3.9). Then, for $t > 0$, a transfer into this portfolio must continuously be made as indicated by (A3.7)

It is worth noting that the continuous transfers for $t > 0$ disappear if $b = 0$. The reason is that the habit level x then will shrink over time, as can be seen from (A3.2), so that the habit portfolio can be allowed to shrink as well. If both $a = 0$ and $b = 0$, the habit level of consumption is given once and for all, and the habit portfolio will remain constant at x_0/r.

Lastly, consider the optimal equity share in this context. From (A2.1), we find $\alpha'(t) = m$, so that

(A3.10) $$\alpha = m\,\mathcal{Y}/W = m\left[\frac{W - x/(r+a-b)}{W}\right].$$

This completes our proof that Constantinides' results generalize to the case of an Epstein-Zin non-expected preference ordering in continuous time.

A4. Time-varying risk-free rate

For this section, the consumption tax plays no role, so that we assume $\tau = 0$. We start with equation (A1.11), which is a second-order non-linear differential equation in $\eta(r)$. Rather than trying to derive an exact solution, we follow Campbell and Viceira in approximating this function around r^* as follows:

$$\eta \approx \eta^* + \eta^* \ln\eta\,(r - r^*),$$

$$\ln\eta\,(r - r^*) \approx C_0 + C_1(r - r^*).$$

We then readily find

$$\eta'/\eta \approx C_1$$

$$\eta''/\eta \approx C_1^2$$

Substituting this into (A1.11) and ignoring quadratic terms in $(r^* - r)$, we obtain the following equation:

$$0 = \eta^*[1 + C_0 + C_1(r^* - r)] - (1-\varepsilon)r - \theta(r^* - r)C_1 - (1-\varepsilon)(1-\lambda)\alpha^*(r^* - r)$$

$$+ \text{terms independent of } r.$$

This equation must hold as an identity in r, which means that we can use the method of undetermined coefficients to identify the parameters C_0 and C_1. In particular, the coefficients multiplying r must sum to zero. Thus,

(A4.1) $$C_1 = \eta'/\eta = (1-\varepsilon)[1 - (1-\lambda)\alpha^*]/(\eta^* + \theta) \leq (1-\varepsilon)/\eta^*,$$

where $\alpha^* = \frac{\mu^*}{\gamma\sigma_Z^2} - \left(\frac{1-1/\gamma}{1-\varepsilon}\right)\beta\,\eta'/\eta$.