Nominal Rigidities, Government Spending, and Long-Run Policy Trade-Off

Abstract

We introduce a simple government that consumes the income taxes collected from households into a model with sticky prices and nominal wages proposed by Tsuzuki and Inoue (2010), which introduces a constant rate of technological change and a constant rate of money growth. Government consumption spending provides households with utility. Tsuzuki and Inoue (2010) examined whether a monetary policy trade-off exists between stabilizing the welfare-relevant employment gap and curbing inflation in the steady state when the rate of technological change decreases. They showed that if only prices are sticky, there is no monetary policy trade-off. However, if both prices and nominal wages are sticky, a monetary policy trade-off exists. We consider whether a monetary policy trade-off exists between stabilizing the welfare gap and curbing inflation when the government sets the income tax rate optimally (so that it maximizes household utility). If only prices are sticky, no trade-off exists; however, if both prices and nominal wages are sticky, a trade-off exists. We also examine the dynamic property (determinacy of equilibrium) of the model. The equilibrium is indeterminate under the rule of a constant rate of money growth; however, the equilibrium is determinate when a simple Taylor rule is introduced.

JEL classification: E52; O42

Keywords: Government consumption; Money growth; Technological change; New Keynesian Phillips curve; Determinacy of equilibrium

1 Introduction

In the so-called New Keynesian (NK) model, which is an optimizing model with sticky prices (nominal wages), the following three levels of output can be defined:

- Efficient level (y_1): The goods (labor) market is perfectly competitive and prices (nominal wages) are flexible.
- Natural level (y_2): The goods (labor) market is monopolistically competitive and prices (nominal wages) are flexible.
- Actual level (y_3): The goods (labor) market is monopolistically competitive and prices (nominal wages) are sticky.

y_1 is the output achieved under the first-best condition, y_2 is the output achieved under the second-best condition, and y_3 is the actual output. The output gap is defined as the gap between y_2 and y_3, and the welfare-relevant output gap is defined as the gap between y_1 and y_3. In the steady state, y_3 coincides with y_2, that is, there is no output gap; however, y_3 discords from y_2 when a technology shock occurs. The inflation rate is zero in the steady state; however, inflation or deflation may occur during the shock period.

In the standard model where only prices are assumed to be sticky, stabilizing the output gap (welfare-relevant output gap) is consistent with curbing inflation. Therefore, no policy trade-off exists between the two because in a sticky price model, the gap between y_1 and y_2 is constant regardless of whether a technology shock exists, and hence, stabilizing the output gap (welfare-relevant output gap) is equivalent to curbing inflation for policymakers.[1]

Erceg et al. (2000) have shown that in a sticky price and nominal wage setting, there exists a trade-off in simultaneously stabilizing the output gap, the price inflation rate, and the nominal wage inflation rate.

In addition, Blanchard and Galí (2007) have proposed a model with sticky prices and real wages, and have shown that there exists a policy trade-off between stabilizing the welfare-relevant output gap and curbing inflation. In their model, the gap between y_1 and

[1]For details of the policy analysis using the NK model, refer to Woodford (2003) and Galí (2008).

y_2 fluctuates when a technology shock occurs, unlike in the standard model, and hence, a policy trade-off exists.[2]

Tsuzuki and Inoue (2010) have introduced the factors of technological change and money growth into a model with sticky prices and nominal wages. Both these factors can affect the steady-state value of y_3. Therefore, in their model, even in the absence of a technology shock, y_3 does not necessarily coincide with y_2 in the steady state.

Tsuzuki and Inoue (2010) examined whether a policy trade-off exists between stabilizing the welfare-relevant employment gap[3] and curbing inflation in the steady state when a permanent, negative shock to the rate of technological change occurs.

They obtained the following results. If only prices are sticky, a monetary policy trade-off does not exist between stabilizing the welfare-relevant employment gap and curbing inflation; however, if both prices and nominal wages are sticky, a monetary policy trade-off may exist between the two. In their model, stabilizing the price inflation rate is not equivalent to stabilizing the nominal wage inflation rate. Hence a policymaker's attempt to stabilize the price inflation rate will cause fluctuations in the nominal wage rate, possibly resulting in a decrease in the actual level of employment (l_3). As the efficient level of employment (l_1) is constant regardless of whether a shock exists, a decrease in l_3 increases the welfare-relevant employment gap.

We introduce a simple government that consumes the income taxes collected from households into the model proposed by Tsuzuki and Inoue (2010). Government consumption spending provides households with utility. In this study, we do not use employment (output) as a measure of welfare because household utility depends not only on the total consumption of goods but also on the proportion of private consumption to government consumption (i.e., income tax rate). Hence we assume that the government stabilizes the welfare gap (gap between efficient and actual utility) rather than the welfare-relevant

[2]According to Blanchard and Galí (2007), the absence of trade-off between stabilizing the welfare-relevant output gap and curbing inflation is one of the unrealistic features of the standard NK model. They call the equivalence of the stabilization of the welfare-relevant output gap and that of the inflation rate a "divine coincidence."

[3]Tsuzuki and Inoue (2010) do not use output but employment to describe the activity level of the economy. However, this difference in the use of variables is not important because in their model, the output y is simply obtained by multiplying technology z by employment l. Following Tsuzuki and Inoue (2010), we use employment in this study. Whereas Tsuzuki and Inoue (2010) develop the model by using difference equations in their discrete-time model, we develop it by using differential equations in a continuous-time model. This difference is also not important.

employment gap.

We consider whether a monetary policy trade-off exists between stabilizing the welfare gap and curbing inflation in the steady state when the government sets the income tax rate optimally (so that it maximizes household utility). We also examine the dynamic property (determinacy of equilibrium) of the model.

This paper is organized as follows. Section 2 describes the development of an NK model that is extended to allow a constant rate of technological change and a constant rate of money growth. Section 3 examines the effect of a decrease in the rate of technological change on the welfare gap. Section 4 linearizes the model around the steady state and investigates the dynamic property. Section 5 presents the conclusion.

2 An NK model

The proposed model assumes an economy that comprises five types of agents: output aggregators (retailers), firms, labor aggregators (employment agents), households, and policymakers (i.e., the central bank and government, or the consolidated government). The output aggregators combine differentiated goods i ($i \in [0,1]$) produced by the firms into a composite good, and then, sell it to the households under perfect competition. Firm i produces good i under monopolistic competition.[4] The labor aggregators combine differentiated labor services j ($j \in [0,1]$) supplied by the households into a composite labor service, and then, sell it to the firms under perfect competition. Household j supplies labor j to the labor aggregators under monopolistic competition. The policymakers supply money, collect taxes, and consume.

The abstract figure of the model is presented in Fig. 1. (In this figure, the policymakers are omitted.)

[4]This framework in which many firms produce differentiated goods under monopolistic competition is based on the study of Blanchard and Kiyotaki (1987).

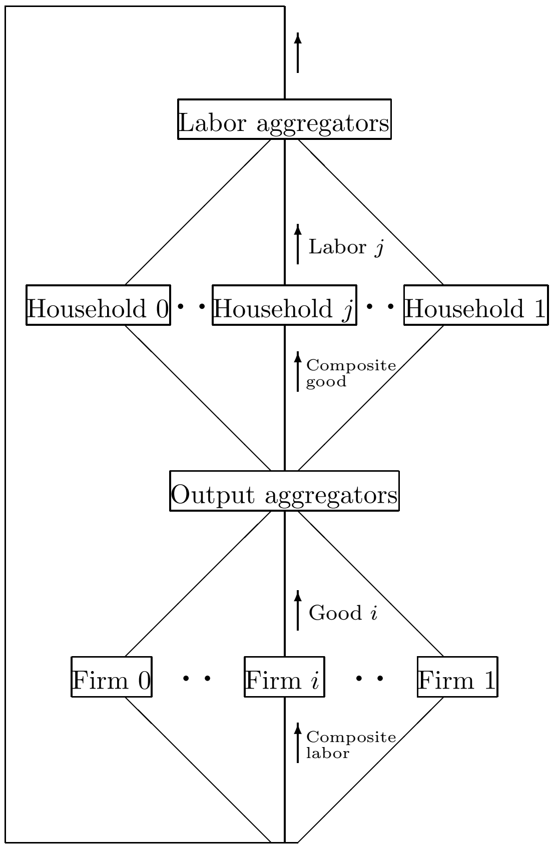

Figure 1: Abstract figure of the model

2.1 Output aggregators

Output aggregators combine the differentiated goods supplied by firms into a composite good. According to the Dixit-Stiglitz formulation[5] (CES function):

$$y = \left[\int_0^1 y_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}, \tag{1}$$

where y is the quantity of the composite good (final good), y_i is the quantity of good i, and ϕ (> 1) is a parameter that represents the elasticity of substitution among the different types of goods.

[5] See Dixit and Stiglitz (1977).

Given the price of good i and the quantity of the composite good that is determined to conform to consumption demand, an output aggregator determines the purchase volume of good i to minimize the cost, which is given by

$$\int_0^1 p_i y_i di, \tag{2}$$

where p_i is the price of good i.

From the first-order conditions for optimality, we obtain

$$y_i = \left(\frac{p_i}{p}\right)^{-\phi} y, \tag{3}$$

(see Appendix A.1) where p is the price of the composite good (general price level), which is given by

$$p = \left[\int_0^1 p_i^{1-\phi} di\right]^{\frac{1}{1-\phi}}. \tag{4}$$

2.2 Labor aggregators

Labor aggregators combine the differentiated labor services supplied by households into a composite labor service. According to the Dixit-Stiglitz formulation:

$$l = \left[\int_0^1 l_j^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}}, \tag{5}$$

where l is the quantity of composite labor, l_j is the amount of labor supplied by household j, and η (> 1) is a parameter that represents the elasticity of substitution among the different types of labor.

Given the nominal wage rate of labor j and the quantity of composite labor that is determined to conform to the demand, a labor aggregator determines the volume of employment of labor j to minimize the cost, which is given by

$$\int_0^1 W_j l_j dj, \tag{6}$$

where W_j is the nominal wage rate of labor j.

From the first-order conditions for optimality, we obtain

$$l_j = \left(\frac{W_j}{W}\right)^{-\eta} l, \tag{7}$$

where W is the nominal wage rate of composite labor, which is given by

$$W = \left[\int_0^1 W_j^{1-\eta} dj \right]^{\frac{1}{1-\eta}}. \tag{8}$$

2.3 Firms

We represent the production function of firm i as follows:

$$y_i = z l_i, \tag{9}$$

where z is the technology and l_i is the quantity of composite labor employed by firm i.

Technology z grows at a constant rate:

$$\frac{\dot{z}}{z} = g, \tag{10}$$

where g (≥ 0) is the rate of technological change.

The real profit of firm i is given by

$$\Pi_i = \frac{p_i y_i - W l_i}{p} - \frac{\gamma}{2}\pi_i^2 y, \tag{11}$$

where $\frac{\gamma}{2}\pi_i^2 y$ is the price adjustment cost,[6] $\pi_i = \frac{\dot{p}_i}{p_i}$, and γ is the scale of the adjustment cost. If $\gamma \to 0$, the price is flexible, but if $\gamma > 0$, the price is sticky. Following Schmitt-Grohé and Uribe (2001), we assume that the price adjustment cost is not the resource cost but the payment made to other agents. This assumption makes the goods-market equilibrium condition simpler, and then, simplifies the development of the model.[7]

Schmitt-Grohé and Uribe (2001) assume that the adjustment cost is paid to the financial institutions. The financial institutions share the received adjustment cost with households, who are assumed to be the owners of the financial institutions.[8] In this study, for the sake of simplicity, we assume that the adjustment cost is directly paid to households. For example, it can be considered to be the provision of a discount coupon for goods for price change notification.

[6]Following Rotemberg (1982), we specify the adjustment cost function in the quadratic form.

[7]Inoue and Tsuzuki (2011) developed an NK model in which the factor of technological change is introduced, as we did in this study, on the assumption that the price adjustment cost is the resource cost. However, they did not consider the wage stickiness.

[8]More appropriately, only a fraction (α) of the price adjustment cost is the true resource cost (i.e., spending of resources), and the remainder is household income. Our model considers the case where $\alpha = 0$.

Given the demand curve for good i as shown in Eq. (3), firm i determines the value of π_i to maximize the discounted present value of a stream of profits, which is given by

$$V_i(t) = \int_0^\infty \left[\frac{p_i(t)y_i(t) - W(t)l_i(t)}{p(t)} - \frac{\gamma}{2}\pi_i(t)^2 y(t)\right] e^{-\int_0^t r(s)ds} dt, \tag{12}$$

where r is the real interest rate.

From the first-order conditions for optimality, we obtain

$$\dot{\pi} + \pi\frac{\dot{y}}{y} = r\pi + \frac{\phi - 1}{\gamma} - \frac{\phi}{\gamma}\frac{w}{z}, \tag{13}$$

(see Appendix A.2), where w is the real wage rate, defined as $w = \frac{W}{p}$. Eq. (13) represents the so-called NK Phillips curve (NKPC).

2.4 Households

Household j obtains utility from consumption c, real cash balances m, and government consumption G, and disutility from labor l_j at every period.[9] We represent the utility function of household j as follows:

$$\ln c + \sigma \ln m + \beta \ln G - \nu \frac{l_j^{1+\psi}}{1+\psi}, \tag{14}$$

where σ (> 0) is the weight of the utility of money holdings, β (> 0) is the weight of the utility of government consumption (substitutability between private and government consumption[10]), ν (> 0) is the weight of the disutility of labor, and ψ (> 0) is the elasticity of the marginal disutility of labor.

Household j possesses nominal cash balances M and shares of stock (which we normalize to unity). The market price of shares is Q, and the shares earn dividend D. Household j receives wage income $W_j l_j$ and the price adjustment cost pn from firms, and transfer income pv from the government.

Each household is assumed to have monopoly power in the labor market, and hence, can determine the nominal wage rate. However, when a household changes the nominal wage rate, it incurs a wage adjustment cost, which is given by $\frac{\delta}{2}\omega_j^2 py$, where δ is the scale

[9]To be precise, c must be written as c_j and m must be written as m_j. However, all households are identical with regard to consumption and money holdings; therefore, we simply denote them as c and m. The connotations of A and D are similar.

[10]If $\beta = 1$, government and private consumption are perfectly substitutable.

of the adjustment cost and $\omega_j = \frac{\dot{W}_j}{W_j}$.[11][12] If $\delta \to 0$, the nominal wage rate is flexible, but if $\delta > 0$, it is sticky. We assume that the wage adjustment cost is not the resource cost but the household payment made to other agents, as is the price adjustment cost. In this study, we assume that the wage adjustment cost is the payment made to the labor unions. For example, it can be considered as a cost of procedures associated with an income change. The labor unions share the received adjustment cost with households as benefit pb.[13]

The government collects income taxes from households and consumes them:

$$G = \tau y, \tag{15}$$

where τ $(0 < \tau < 1)$ is the income tax rate.[14]

Let A denote the household's nominal assets, and the following equation holds:

$$\dot{A} = \dot{Q} + D + W_j l_j + pn + pv + pb - \frac{\delta}{2}\omega_j^2 py - pc - p\tau y. \tag{16}$$

[11]In this specification, the wage adjustment cost is proportional to the income py. We can also specify it as being proportional to the wage Wl. For ease of calculation, we adopt the former assumption: the wage adjustment cost is proportional to py. The main results do not depend on these specifications.

[12]The findings of this study and those of Tsuzuki and Inoue (2010) are strongly dependent on the function form of the adjustment cost functions. If we formulate the price adjustment cost as $\frac{\gamma}{2}(\pi_i - \pi^*)^2 y$ not as $\frac{\gamma}{2}\pi_i^2 y$, as well as the wage adjustment cost as $\frac{\delta}{2}(\omega_j - \omega^*)^2 py$ not as $\frac{\delta}{2}\omega_j^2 py$, the actual level of employment will accord with the natural level of employment in the steady state regardless of the rate of money growth; that is, there exists the super-neutrality of money. The formulations of $\frac{\gamma}{2}\pi_i^2 y$ and $\frac{\delta}{2}\omega_j^2 py$ imply that changes in prices and wages incur adjustment costs even when the changes are very small. On the other hand, the formulations of $\frac{\gamma}{2}(\pi_i - \pi^*)^2 y$ and $\frac{\delta}{2}(\omega_j - \omega^*)^2 py$ imply that changes in prices and wages incur adjustment costs only when the magnitude of the changes exceeds the steady-state values of the inflation rate and wage inflation rate.

[13]On this assumption, the wage adjustment cost exists in the household budget constraint. Another assumption can be that the wage adjustment cost exists in the utility function (e.g., Benhabib et al. 2001). In this case, Eq. (14) must be written as $\ln c + \sigma \ln m + \beta \ln G - \nu \frac{l_j^{1+\psi}}{1+\psi} - \frac{\delta}{2}\omega_j^2$. However, the wage version of the NKPC that is derived from a household optimization problem would be almost the same in each case. In addition, the wage adjustment cost existing in the utility function does not incur resource cost, and hence, we need not to assume that it is payment made to other agents, to simplify the goods-market equilibrium condition.

[14]Even if we consider the consumption tax rather than the income tax as government revenues, the results obtained would not be qualitatively changed. In the case of the consumption tax, Eq. (15) is replaced by $G = \tau c$.

As the household's nominal assets are composed of cash and stocks, $A = M + Q$. In addition, using the capital gain $\dot{Q}$ and the dividend D, the rate of return on shares E can be written as $E = \frac{\dot{Q}+D}{Q}$. As a result of arbitrage, the rate of return on shares should be equal to the nominal interest rate R $(= r + \pi)$: $R = E$. Using these equations, Eq. (16) can be rewritten as follows:

$$\dot{a} = ra + w_j l_j + n + v + b - \frac{\delta}{2}\omega_j^2 y - c - \tau y - Rm, \tag{17}$$

where a is the real value of assets, defined as $a = \frac{A}{p}$.

Given the demand curve for labor j as shown in Eq. (7) and the budget constraint given by Eq. (17), household j determines the values of c, m, and ω_j to maximize the discounted present value of a stream of utility, which is given by

$$U_j = \int_0^\infty \left[\ln c + \sigma \ln m + \beta \ln G - \nu \frac{l_j^{1+\psi}}{1+\psi} \right] e^{-\rho t} dt, \tag{18}$$

where ρ (> 0) is the subjective discount rate.

From the first-order conditions for optimality, we obtain

$$\frac{\dot{c}}{c} + \pi + \rho = R = \sigma \frac{c}{m}, \tag{19}$$

$$\frac{\dot{\omega}}{\omega} = \rho + \frac{\dot{c}}{c} - \frac{\dot{y}}{y} - \frac{\nu\eta}{\delta} l^{1+\psi} \frac{c}{\omega y} + \frac{\eta - 1}{\delta} \frac{w}{z} \frac{1}{\omega}, \tag{20}$$

(see Appendix A.3). Eq. (20) represents the wage version of the NKPC.

2.5 Money supply

The central bank increases money supply M at a constant rate:

$$\frac{\dot{M}}{M} = \theta, \tag{21}$$

where θ (≥ 0) is the growth rate of money.

As real cash balances are defined as $m = \frac{M}{p}$, the growth rate of real cash balances is given by

$$\frac{\dot{m}}{m} = \frac{\dot{M}}{M} - \frac{\dot{p}}{p} = \theta - \pi. \tag{22}$$

2.6 Goods-market equilibrium condition

We assume that the transfer expenditure pv by the government is financed by seigniorage: $pv = \dot{M}$. The price and wage adjustment costs can be stated in nominal terms as $pn = \frac{\gamma}{2}\pi^2 py$ and $pb = \frac{\delta}{2}\omega^2 py$, respectively. As the dividend D is equivalent to the firms' profit $p\Pi$, $D = p(y - \frac{Wl}{p} - \frac{\gamma}{2}\pi^2 y)$.

Substituting these equations and Eq. (15) into the households' budget constraint given by Eq. (16), we obtain

$$y = c + G, \tag{23}$$

which is the goods-market equilibrium condition.

3 Steady state

The equations presented in the previous section can be considered together as comprising a system of differential equations, as shown below (see Appendix A.4):

$$\dot{R} = (R - \theta - \rho)R, \tag{24}$$

$$\dot{l} = (R - \rho - \pi - g)l, \tag{25}$$

$$\dot{\pi} = \rho\pi + \frac{\phi - 1}{\gamma} - \frac{\phi}{\gamma}s, \tag{26}$$

$$\dot{\omega} = \rho\omega - \frac{\nu\eta}{\delta}l^{1+\psi}(1 - \tau) + \frac{\eta - 1}{\delta}s, \tag{27}$$

$$\dot{s} = (\omega - \pi - g)s, \tag{28}$$

where $s = \frac{w}{z}$.

The nontrivial solutions of this system are given by

$$R^* = \theta + \rho, \tag{29}$$

$$l^* = \left[\frac{\delta\phi\rho\omega^* + \gamma\rho\pi^*(\eta - 1) + (\phi - 1)(\eta - 1)}{\phi\nu\eta(1 - \tau)}\right]^{\frac{1}{1+\psi}}, \tag{30}$$

$$\pi^* = \theta - g, \tag{31}$$

$$\omega^* = \theta, \tag{32}$$

$$s^* = \frac{\gamma\rho\pi^* + \phi - 1}{\phi}. \tag{33}$$

3.1 Case where only prices are sticky

When we assume that prices are sticky but nominal wages are not, the efficient (l_1^*), the natural (l_2^*), and the actual (l_3^*) levels of employment are given by[15]

$$l_1^* = \left[\frac{1}{\nu(1-\tau)}\right]^{\frac{1}{1+\psi}} \quad (\phi \to \infty, \gamma \to 0, \eta \to \infty, \delta \to 0), \tag{34}$$

$$l_2^* = \left[\frac{\phi - 1}{\phi\nu(1-\tau)}\right]^{\frac{1}{1+\psi}} \quad (\phi \in (1, \infty), \gamma \to 0, \eta \to \infty, \delta \to 0), \tag{35}$$

$$l_3^* = \left[\frac{\gamma\rho(\theta - g) + \phi - 1}{\phi\nu(1-\tau)}\right]^{\frac{1}{1+\psi}} \quad (\phi \in (1, \infty), \gamma > 0, \eta \to \infty, \delta \to 0). \tag{36}$$

Using Eqs. (9), (15), (19), and (23), c, m, and G can be written as follows:

$$c = (1-\tau)zl, \tag{37}$$

$$\begin{aligned} m &= z\frac{c}{z}\frac{m}{c} \\ &= z(1-\tau)l\frac{\sigma}{R}, \end{aligned} \tag{38}$$

$$G = \tau zl. \tag{39}$$

The steady-state utility is given from Eqs. (14), (37), (38), and (39) by

$$\begin{aligned} u^* = &(1+\sigma)\ln(1-\tau) + (1+\sigma+\beta)\ln z + (1+\sigma+\beta)\ln l^* \\ &+ \sigma\ln\sigma - \sigma\ln R^* + \beta\ln\tau - \nu\frac{(l^*)^{1+\psi}}{1+\psi}. \end{aligned} \tag{40}$$

As the terms involving $\ln z$, $\ln\sigma$, and $\ln R^*$ do not affect the welfare gap (gap between efficient and actual utility), we henceforth omit them.

Substituting Eq. (34) into Eq. (40), we obtain the efficient utility u_1^*. Similarly, substituting Eq. (35) into Eq. (40), we obtain the natural utility u_2^*, and substituting Eq. (36) into Eq. (40), we obtain the actual utility u_3^*.

u_1^* and u_2^* are independent of g because l_1^* and l_2^* are independent of g. However, u_3^* is dependent on g because l_3^* is dependent on g. When $\theta = g$, however, $l_2^* = l_3^*$ and $u_2^* = u_3^*$, and therefore, u_3^* is also independent of g .

[15]The assumption $\eta \to \infty$ implies that all laborers are homogeneous (substitutable); that is, the labor market is perfectly competitive. In addition, the assumption $\delta \to 0$ implies that the wage adjustment cost is zero; that is, the nominal wage rate is flexible. The connotations of ϕ and γ are similar.

The government determines the income tax rate τ to maximize u^*. Let τ_1^*, τ_2^*, and τ_3^* be the tax rates that maximize u_1^*, u_2^*, and u_3^*, respectively. These rates must satisfy the first-order condition for utility maximization:

$$\frac{\partial u^*}{\partial \tau} = -\frac{1+\sigma}{1-\tau} + \frac{\beta}{\tau} + \{(1+\sigma+\beta)l^*(\tau)^{-1} - \nu l^*(\tau)^{\psi}\}l^{*\prime}(\tau) = 0, \tag{41}$$

and the second-order condition:

$$\begin{aligned}\frac{\partial^2 u^*}{\partial \tau^2} = &-\frac{1+\sigma}{(1-\tau)^2} - \frac{\beta}{\tau^2} + \{(1+\sigma+\beta)(l^*)^{-1} - \nu(l^*)^{\psi}\}l^{*\prime\prime}(\tau) \\ &- \{(1+\sigma+\beta)(l^*)^{-2} + \psi\nu(l^*)^{\psi-1}\}l^{*\prime}(\tau)^2 < 0.\end{aligned} \tag{42}$$

We henceforth assume that both of these conditions are satisfied.

We assume that the central bank adopts the policy that curbs inflation in the steady state ($\pi^* = 0$):

$$\theta = g. \tag{43}$$

Setting the parameters at reasonable values, we examine the relations between the income tax rate τ and the utility u^* by means of computational simulations. Following Fujiwara (2010), we assume that $\phi = 6$, $\gamma = 27$, $\rho = 0.01$, $\psi = 1$, $\nu = 109.82$, and $g = 0.03$. As the literature on government spending estimates β to be between 0.2–0.5, we assume that $\beta = 0.3$. In addition, the effect of money holdings on utility (share of the utility of money holdings in the total utility) is much smaller than that of consumption (see, e.g., Farmer 1997). Following the literature, we assume that $\sigma = 0.00016$.

Fig. 2 shows the results of the simulation study. The income tax rate τ is on the horizontal axis and the utility u^* is on the vertical axis. The solid line is u_3^* $(= u_2^*)$ and the dotted line is u_1^*. The figure shows that the utility level achieved by the optimal tax rate τ_3^*, which corresponds to the maximum value of u_3^*, is less than that by τ_1^*, which corresponds to the maximum value of u_1^*. The gap between these two levels is the welfare gap.

Let us consider a situation where the rate of technological change g decreases. The central bank can continue to curb inflation by decreasing the rate of money growth θ to sustain $\theta = g$ (that is, $\dot{\theta} = \dot{g}$). Under this policy, u_3^* is independent of g, and therefore, the welfare gap does not change even when g decreases. Thus, in this case, there is no monetary policy trade-off between stabilizing the welfare gap and curbing inflation.

The results can be summarized as follows:

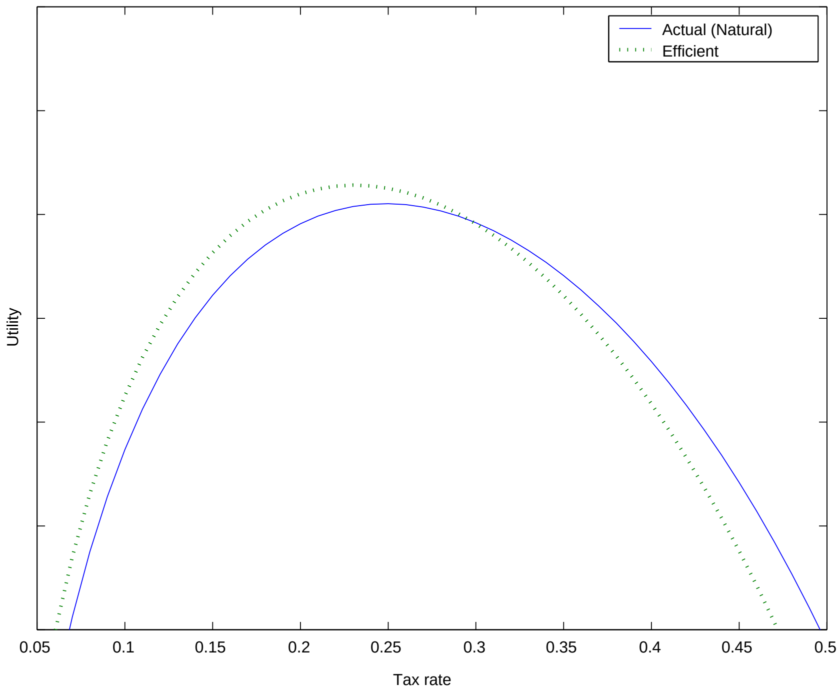

Figure 2: Income tax rate and utility

Proposition 1 *If the labor market is perfectly competitive and nominal wages are flexible, but the goods market is monopolistically competitive and prices are sticky, there is no monetary policy trade-off between stabilizing the welfare gap and curbing inflation in the steady state as long as the government sets the income tax rate at the optimal rate.*

Proposition 1 validates the findings of Tsuzuki and Inoue (2010). Their model corresponds to the case where $\beta = 0$ in our model.

On the assumption that the government sets the income tax rate at the optimal rate, the central bank can achieve the maximum value of u_1^* by adopting a policy that causes inflation: $\theta > g$. Fig. 3 shows the relation between the rate of money growth θ and the utility u^* under the optimal taxation policy (where the effect of g on u^* through the term $\sigma \ln R^*$ is considered). The rate of money growth θ is on the horizontal axis and the utility u^* is on the vertical axis. The solid line is u_3^* and the dotted line is u_1^*. u_3^* coincides with u_1^* when θ takes a certain value lying between 3 and 4. However, such a value of θ (rate of money growth of 300–400%) is unrealistic.

As shown by Fig. 3, u_1^* decreases with increasing θ. The cause of this is related to the fifth term of Eq. (40): $\sigma \ln R^*$. That is, an increase in the rate of technological change

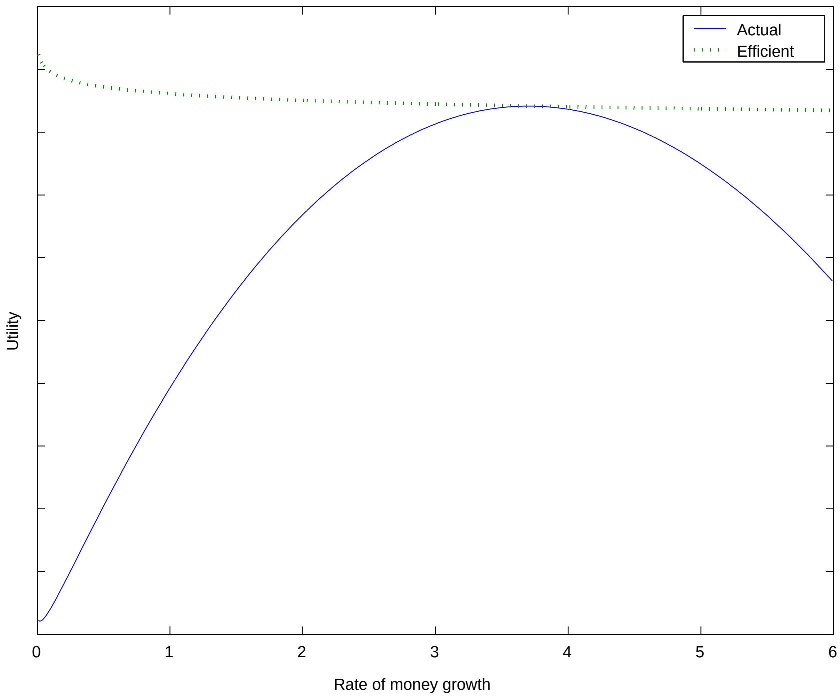

Figure 3: Rate of moeny growth and utility

g decreases the utility u^* through the following process: increase in the rate of money growth $\theta \longrightarrow$ increase in the nominal interest rate $R^* \longrightarrow$ decrease in the real cash balances $m^* \longrightarrow$ decrease in the utility u^*. The term $\sigma \ln R^*$ has the same effect on u_3^*, but it is negated by an opposite effect brought by an increase in the employment l_3^*.

For any values of θ, the central bank can stabilize the welfare gap by adopting the policy $\dot{\theta} = \dot{g}$ because this policy does not change l_3^* ($\frac{\partial l_3^*}{\partial \theta} + \frac{\partial l_3^*}{\partial g} = 0$). Therefore, there exists no monetary policy trade-off between stabilizing the welfare gap and "stabilizing" the inflation rate.

3.2 Case where both prices and nominal wages are sticky

When we assume that both prices and nominal wages are sticky, the efficient, the natural, and the actual levels of employment are given by

$$l_1^* = \left[\frac{1}{\nu(1-\tau)}\right]^{\frac{1}{1+\psi}} \quad (\phi \to \infty, \gamma \to 0, \eta \to \infty, \delta \to 0), \tag{44}$$

$$l_2^* = \left[\frac{(\phi-1)(\eta-1)}{\phi\nu\eta(1-\tau)}\right]^{\frac{1}{1+\psi}} \quad (\phi \in (1,\infty), \gamma \to 0, \eta \in (1,\infty), \delta \to 0), \tag{45}$$

$$l_3^* = \left[\frac{\delta\phi\rho\theta + \gamma\rho(\theta - g)(\eta-1) + (\phi-1)(\eta-1)}{\phi\nu\eta(1-\tau)}\right]^{\frac{1}{1+\psi}}$$
$$(\phi \in (1,\infty), \gamma > 0, \eta \in (1,\infty), \delta > 0). \tag{46}$$

Consider a situation where, in the actual economy, the central bank adopts the policy $\theta = g$ to curb inflation in the steady state ($\pi^* = 0$), and the government sets the income tax rate at the optimal rate τ_3^*. Unlike in the previous section, $l_2^* \neq l_3^*$ and $u_2^* \neq u_3^*$ even when $\theta = g$. Therefore, u_3^* is dependent on g even when $\theta = g$.

We set the parameters at reasonable values following Farmer (1997) and Fujiwara (2010): $\phi = 6$, $\gamma = 27$, $\rho = 0.01$, $\psi = 1$, $\nu = 109.82$, $\beta = 0.3$, $\sigma = 0.00016$, $\eta = 21$, and $\delta = 199$. In addition, we omit the terms involving $\ln z$, $\ln \sigma$, and $\ln R^*$, as in the previous section.

Fig. 4 shows the results of the simulation study. The income tax rate τ is on the horizontal axis and the utility u^* is on the vertical axis. The solid line is u_3^* when $g = 0.03$, the dashed line is u_3^* when $g = 0$, and the dotted line is u_1^*.

When $g = 0.03$, the utility level achieved by τ_3^* is less than that by τ_1^*. Furthermore, when $g = 0$, the utility level achieved by τ_3^* is less than that when $g = 0.03$. Therefore, the welfare gap increases when g decreases.

Moreover, although it is hard to see in the figure, the value of τ_3^* when $g = 0$ is a little greater than that when $g = 0.03$. This implies that the government must increase the income tax rate τ when g decreases; that is, $\frac{\partial \tau_3^*}{\partial g} < 0$.

Thus, if the central bank continues to curb inflation when g decreases, the welfare gap increases.

On the other hand, if the central bank stabilizes the welfare gap, it must adopt the policy $\dot{\theta} > \dot{g}$ and tolerate inflation. Therefore, in this case, there exists a monetary policy trade-off between stabilizing the welfare gap and curbing inflation.

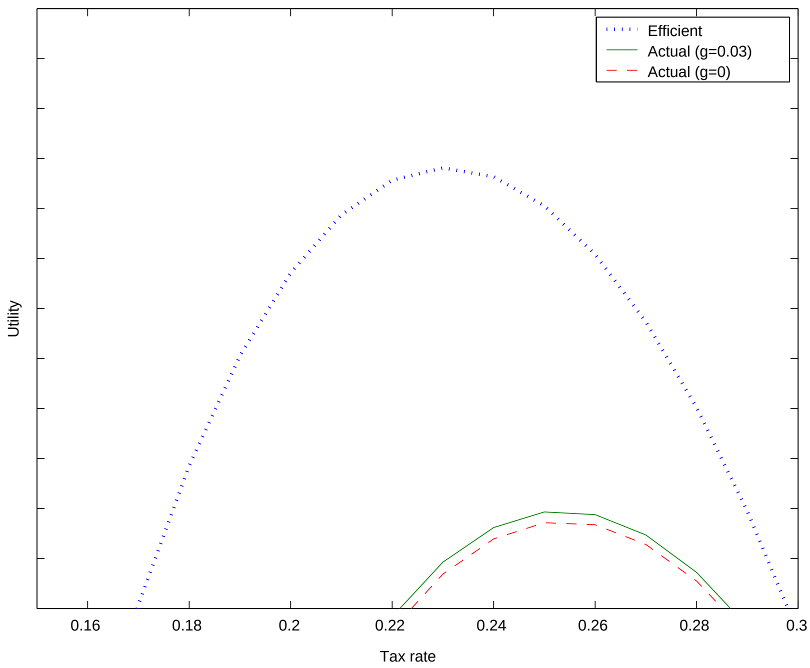

Figure 4: Income tax rate and utility

The results can be summarized as follows:

Proposition 2 *If both the goods and the labor markets are monopolistically competitive and both prices and nominal wages are sticky, there exists a monetary policy trade-off between stabilizing the welfare gap and curbing inflation in the steady state even when the government sets the income tax rate at the optimal rate.*

This also corresponds with the findings of Tsuzuki and Inoue (2010).[16]

In the case where both prices and nominal wages are sticky, the rate of money growth needed for achieving u_1^* is about 160% (where $g = 0.03$). This rate is much smaller than that in the case where only prices are sticky (300–400%), but it is still unrealistic value.

[16]Tsuzuki and Inoue (2011b) showed that this trade-off is dependent on the definition of the nominal wage rate; that is, if nominal wage rates per unit of effective labor, rather than that of labor are sticky, there is no policy trade-off.

4 Determinacy of equilibrium

In this section, we examine the dynamic property of the system comprised of Eqs. (24)–(28). The dynamic property is characterized by a concept of stability called the "determinacy of equilibrium." This concept is defined using the number of non-predetermined variables and the number of divergent roots (roots with positive real parts) in the relevant system as follows:

1. The number of non-predetermined variables = the number of divergent roots → Determinate

2. The number of non-predetermined variables > the number of divergent roots → Indeterminate

3. The number of non-predetermined variables < the number of divergent roots → No equilibrium path

The non-predetermined variables are defined as those whose initial values are not given, or those whose values may jump discontinuously. For example, the inflation rate π and the wage inflation rate ω are non-predetermined variables in our model.[17]

If an equilibrium is determinate, the initial values of non-predetermined variables that lead the solutions to the steady state exist uniquely. On the other hand, if an equilibrium is indeterminate, the initial values of non-predetermined variables that lead the solutions to the steady state exist innumerably. In the case of no equilibrium path, the solutions do never converge to the steady state in any choice of the initial values of non-predetermined variables (except in cases where the initial values just exist in the steady state).

A determinate equilibrium is regarded as a desirable (stable) one because the path leading the economy to the steady state (equilibrium path) exists uniquely. On the other hand, an indeterminate equilibrium is regarded as an unstable one because sunspot fluctuations may arise.[18] In cases where there is no equilibrium path, the equilibrium is regarded as an unstable one.

[17] The control variables of a dynamic optimization problem are definitely the non-predetermined variables.

[18] Indeterminacy of equilibrium is a necessary condition for the existence of sunspot fluctuations, which are the economic fluctuations induced by changes in expectation. See Carlstrom and Fuerst (2001), Bullard and Mitra (2002), Carlstrom and Fuerst (2007), and Tsuzuki and Inoue (2011a) for details.

First, we investigate which of the above three cases applies to our model. Next, we introduce a rule under which the central bank controls the nominal interest rate in response to changes in the inflation rate, in place of the rule of a constant rate of money growth.

4.1 Case where the rate of money growth is constant

The Jacobian matrix of the system comprised of Eqs. (24)–(28) evaluated at the steady state is given by

$$J_1 = \begin{bmatrix} R^* & 0 & 0 & 0 & 0 \\ l^* & 0 & -l^* & 0 & 0 \\ 0 & 0 & \rho & 0 & -\frac{\phi}{\gamma} \\ 0 & -(1+\psi)\frac{\eta}{\delta}l^{*\psi}(1-\tau) & 0 & \rho & \frac{\eta-1}{\delta} \\ 0 & 0 & -s^* & s^* & 0 \end{bmatrix}. \quad (47)$$

The matrix J_1 is decomposable and the characteristic equation can be written as follows:

$$\Delta_1(x) = (R^* - x)\begin{vmatrix} -x & -l^* & 0 & 0 \\ 0 & \rho - x & 0 & -\frac{\phi}{\gamma} \\ -(1+\psi)\frac{\eta}{\delta}l^{*\psi}(1-\tau) & 0 & \rho - x & \frac{\eta-1}{\delta} \\ 0 & -s^* & s^* & -x \end{vmatrix} = 0, \quad (48)$$

which shows that one of the characteristic roots of J_1 is a positive real number (R^*).

The standard equation used to obtain the remaining four roots is given by

$$\Delta_2(x) = \begin{vmatrix} -x & -l^* & 0 & 0 \\ 0 & \rho - x & 0 & -\frac{\phi}{\gamma} \\ -(1+\psi)\frac{\eta}{\delta}l^{*\psi}(1-\tau) & 0 & \rho - x & \frac{\eta-1}{\delta} \\ 0 & -s^* & s^* & -x \end{vmatrix} = 0. \quad (49)$$

We can expand this determinant by the first row to obtain

$$(\rho - x)^2 x^2 + \left\{\frac{\phi s^*}{\gamma} - \frac{(\eta-1)s^*}{\delta}\right\}(\rho - x)x + \frac{(1+\psi)\eta(l^*)^{1+\psi}(1-\tau)s^*\phi}{\gamma\delta}. \quad (50)$$

Hence, Eq. (49) can be rewritten as follows:

$$\Delta_2(x) = o^2 + \left\{\frac{\phi s^*}{\gamma} - \frac{(\eta-1)s^*}{\delta}\right\}o + \frac{(1+\psi)\eta(l^*)^{1+\psi}(1-\tau)s^*\phi}{\gamma\delta} = 0, \quad (51)$$

where $o = (\rho - x)x$.

Denoting the roots of Eq. (51) as o_1 and o_2, we obtain

$$o_1, o_2 = \frac{-\{\frac{\phi s^*}{\gamma} + \frac{(\eta-1)s^*}{\delta}\} \pm \sqrt{\{\frac{\phi s^*}{\gamma} + \frac{(\eta-1)s^*}{\delta}\}^2 - 4\frac{(1+\psi)\eta(l^*)^{1+\psi}(1-\tau)s^*\phi}{\gamma\delta}}}{2}, \quad (52)$$

which shows that the real parts of both o_1 and o_2 are negative.

As $o = (\rho - x)x$, we obtain

$$x^2 - \rho x + o_1 = 0, \quad (53)$$

$$x^2 - \rho x + o_2 = 0. \quad (54)$$

Denoting the roots of Eq. (53) as x_1 and x_2, and the roots of Eq. (54) as x_3 and x_4, we obtain

$$x_1, x_2 = \frac{\rho \pm \sqrt{\rho^2 - 4o_1}}{2}, \quad (55)$$

$$x_3, x_4 = \frac{\rho \pm \sqrt{\rho^2 - 4o_2}}{2}. \quad (56)$$

As the real parts of both o_1 and o_2 are negative, x_i $(i = 1, 2, 3, 4)$ contains two roots with positive real parts and two roots with negative real parts.

Thus, the total number of divergent roots is three. On the other hand, the number of non-predetermined variables of the system comprised of Eqs. (24)–(28) is four: R, l, π, and ω. Therefore, we can conclude that the equilibrium is indeterminate.

4.2 Case where a simple Taylor rule is introduced

In this section, we consider a case where a simple Taylor rule, under which the central bank controls the nominal interest rate in response to changes in the inflation rate, is introduced in place of the rule of a constant rate of money growth as represented by $\frac{\dot{M}}{M} = \theta$.

A simple Taylor rule whose target variable is only the inflation rate can be represented by

$$R - R^* = \varepsilon_a(\pi - \pi^*), \quad (57)$$

or

$$R = R^* e^{\frac{\varepsilon_b}{R^*}(\pi - \pi^*)}, \quad (58)$$

where ε_a (> 0) denotes the degree of response of R to changes in π and ε_b (> 0) denotes the elasticity of R with respect to π. Both Eqs. (57) and (58) imply that the central bank sets the nominal interest rate at a level higher (lower) than the steady-state value (targeted value) when the inflation rate exceeds (falls below) the steady-state value.

Eq. (58) is the Taylor rule with the zero bound on nominal interest rates proposed by Benhabib et al. (2001). They showed that there exists a steady-state value of π other than the targeted value π^* when $\varepsilon_b \neq 1$. In the following discussion, we adopt the normal type of the rule represented by Eq. (57).

As shown by Eq. (19) ($R = \sigma \frac{c}{m}$), the nominal interest rate R has a simple inverse relationship with the stock of real cash balances m, and hence, it is not important whether we adopt R or M as the control variable of a policy rule.

Using Eq. (57) in place of Eq. (24), the element in the first row and the second column of the determinant of Eq. (49) ($-l^*$) is replaced by $(\varepsilon_a - 1)l^*$. There is no other changes. In this case, the characteristic equation $\Delta_2(x)$ is given by

$$o^2 + \left\{ \frac{\phi s^*}{\gamma} - \frac{(\eta - 1)s^*}{\delta} \right\} o + \frac{(1-\varepsilon_a)(1+\psi)\eta(l^*)^{1+\psi}(1-\tau)s^*\phi}{\gamma\delta} = 0, \tag{59}$$

and

$$o_1, o_2 = \frac{-\{\frac{\phi s^*}{\gamma} + \frac{(\eta-1)s^*}{\delta}\} \pm \sqrt{\{\frac{\phi s^*}{\gamma} + \frac{(\eta-1)s^*}{\delta}\}^2 - 4\frac{(1-\varepsilon_a)(1+\psi)\eta(l^*)^{1+\psi}(1-\tau)s^*\phi}{\gamma\delta}}}{2}. \tag{60}$$

If $\varepsilon_a < 1$, the result is the same as in the previous case; that is, the real parts of both o_1 and o_2 are negative, and hence, Eqs. (55) and (56) demonstrate that x_i ($i = 1, 2, 3, 4$) contains two roots with positive real parts and two roots with negative real parts. Thus, the number of divergent roots is two. The number of non-predetermined variables is three: l, π, and ω. Therefore, we can conclude that if $\varepsilon_a < 1$, the equilibrium is indeterminate.

However, if $\varepsilon_a > 1$, then o_1 and o_2 are positive and negative real roots. In this case, x_i ($i = 1, 2, 3, 4$) contains three roots with positive real parts and one root with negative real part. Thus, the number of divergent roots is three. Therefore, we can conclude that if $\varepsilon_a > 1$, the equilibrium is determinate.

The condition $\varepsilon_a > 1$ is the "Taylor principle," which is well known as a necessary condition for determinacy in NK models.

5 Conclusion

We examined a monetary policy trade-off in the steady state demonstrated by Tsuzuki and Inoue (2010), using an NK model that introduced a simple government.

We set the parameters at reasonable values and did computer simulations. Our results basically correspond with the findings of Tsuzuki and Inoue (2010).

If only prices are sticky, a monetary policy trade-off does not exist between stabilizing the welfare gap and curbing inflation as long as the government sets the income tax rate at the optimal rate.

However, if both prices and nominal wages are sticky, a monetary policy trade-off exists even when the government sets the income tax rate at the optimal rate.

In addition, out of the steady state, the central bank must follow a discretionary policy rule, such as the Taylor rule. Otherwise the equilibrium will be indeterminate.

Acknowledgments

We have received helpful comments from Professor Toichiro Asada (Chuo University), Professor Akio Matsumoto (Chuo University), Professor Hiroyuki Yoshida (Nihon University), Professor Kenichi Ishiyama (Kokushikan University), Professor Hirokazu Takizawa (Chuo University), and participants in a seminar at Chuo University.

A Appendix

A.1 Output aggregator's optimization problem

The representative output aggregator's optimization problem can be summarized as follows:[19]

$$\min_{y_i} \int_0^1 p_i y_i di, \tag{A.1}$$

$$\text{subject to} \quad \left[\int_0^1 y_i^{\frac{\phi-1}{\phi}} di\right]^{\frac{\phi}{\phi-1}} = y. \tag{A.2}$$

[19]This type of optimization problem with an equality integral constraint is known as the "isoperimetric problem."

We derive Eqs. (3) and (4) by solving this problem.

The Lagrangian function of this problem is defined as follows:

$$\mathcal{L} = -\int_0^1 p_i y_i di + \lambda(y^{\frac{\phi-1}{\phi}} - \int_0^1 y_i^{\frac{\phi-1}{\phi}} di), \tag{A.3}$$

where λ is the Lagrange multiplier. The first-order condition for optimality is given by[20]

$$\frac{\partial \mathcal{L}}{\partial y_i} = -p_i - \lambda \frac{\phi-1}{\phi} y_i^{-\frac{1}{\phi}} = 0. \tag{A.4}$$

This equation can be rewritten as follows:

$$y_i = \left\{ \frac{\phi}{\lambda(1-\phi)} \right\}^{-\phi} p_i^{-\phi}. \tag{A.5}$$

As the output aggregators are working under perfect competition, there is no profit; that is, the total revenue is equal to the total cost. The total revenue is py, and the total cost is given by Eq. (2). Therefore, the following equation must hold in the equilibrium:

$$py = \int_0^1 p_i y_i di. \tag{A.6}$$

Substituting Eq. (A.5) into Eqs. (A.2) and (A.6), we obtain

$$\left\{ \frac{\phi}{\lambda(1-\phi)} \right\}^{-\phi} \left[\int_0^1 p_i^{1-\phi} di \right]^{\frac{\phi}{\phi-1}} = y, \tag{A.7}$$

$$py = \left\{ \frac{\phi}{\lambda(1-\phi)} \right\}^{-\phi} \int_0^1 p_i^{1-\phi} di. \tag{A.8}$$

Substituting Eq. (A.7) into Eq. (A.8), we obtain

$$p = \left[\int_0^1 p_i^{1-\phi} di \right]^{\frac{1}{1-\phi}}, \tag{A.9}$$

which is Eq. (4).

Using Eq. (A.9), Eq. (A.7) can be rewritten as follows:

$$\left\{ \frac{\phi}{\lambda(1-\phi)} \right\}^{-\phi} p^{-\phi} = y. \tag{A.10}$$

[20] As $p_i > 0$ and $y_i > 0$, $\lambda < 0$, and hence, $\frac{\partial^2 \mathcal{L}}{\partial y_i^2} = \frac{1}{\phi} \lambda \frac{\phi-1}{\phi} y_i^{-\frac{1}{\phi}-1} < 0$. Therefore, the second-order condition for maximizing $\mathcal{L}$ is satisfied.

Substituting this equation into Eq. (A.5), we obtain

$$y_i = \left(\frac{p_i}{p}\right)^{-\phi} y, \tag{A.11}$$

which is Eq. (3).

A.2 Firm's optimization problem

Firm i's optimization problem can be summarized as follows:

$$\begin{aligned} \max_{\pi_i} \ V_i &= \int_0^\infty \left[\frac{p_i y_i - W l_i}{p} - \frac{\gamma}{2}\pi_i^2 y\right] e^{-\int_0^t r(s)ds} dt, \\ \text{subject to} \quad y_i &= \left(\frac{p_i}{p}\right)^{-\phi} y, \\ \dot{p}_i &= \pi_i p_i. \end{aligned} \tag{A.12}$$

Considering the production function given by Eq. (9), the Hamiltonian function of this problem is defined as follows:

$$\mathcal{H} = \frac{p_i(\frac{p_i}{p})^{-\phi} y - W(\frac{p_i}{p})^{-\phi}\frac{y}{z}}{p} - \frac{\gamma}{2}\pi_i^2 y + \lambda \pi_i p_i, \tag{A.13}$$

where λ is the co-state variable associated with the state variable p_i. The first-order conditions for optimality are given by

$$\frac{\partial \mathcal{H}}{\partial \pi_i} = -\gamma \pi_i y + \lambda p_i = 0, \tag{A.14}$$

$$\dot{\lambda} = r\lambda - \frac{\partial \mathcal{H}}{\partial p_i} = r\lambda - \{(1-\phi)\frac{y_i}{p} + \frac{W}{p}\phi\frac{y_i}{p_i z} + \lambda\pi_i\}. \tag{A.15}$$

The transversality condition is given by

$$\lim_{t\to\infty} \lambda p_i e^{-\int_0^t r(s)ds} = 0. \tag{A.16}$$

We derive Eq. (13), which represents the NKPC, using Eqs. (A.14) and (A.15).

Derivation of Eq. (13)

The following equations can be obtained from Eq. (A.14):

$$\lambda = \frac{\gamma \pi_i y}{p_i}, \tag{A.17}$$

$$\dot{\lambda} = -\frac{\gamma \pi_i^2 y}{p_i} + \gamma \dot{\pi}_i \frac{y}{p_i} + \gamma \dot{y}\frac{\pi_i}{p_i}. \tag{A.18}$$

Substituting these equations into Eq. (A.15), we obtain

$$-\frac{\gamma\pi_i^2 y}{p_i}+\gamma\dot{\pi}_i\frac{y}{p_i}+\gamma\dot{y}\frac{\pi_i}{p_i}=r\frac{\gamma\pi_i y}{p_i}-(1-\phi)\frac{y_i}{p}-\frac{W}{p}\phi\frac{y_i}{p_i z}-\frac{\gamma\pi_i^2 y}{p_i}. \tag{A.19}$$

As all firms behave similarly in accordance with Eq. (A.19), we can drop subscript i from p_i, π_i, and y_i; that is, $p_i = p$, $\pi_i = \pi$, and $y_i = y$.[21] Taking these equations into account and multiplying both sides of Eq. (A.19) by $\frac{p}{\gamma y}$, we obtain

$$\dot{\pi}+\pi\frac{\dot{y}}{y}=r\pi+\frac{\phi-1}{\gamma}-\frac{\phi}{\gamma}\frac{w}{z}, \tag{A.20}$$

where $w=\frac{W}{p}$.

Transversality condition

We show that the steady state obtained in Section 3 satisfies the transversality condition given by Eq. (A.16).

In the steady state, $r = \rho + g$. In addition, we obtain $\lambda p_i = \gamma\pi_i y$ from Eq. (A.14). Considering $y = zl$ and the assumption that technology z grows at a constant rate g, we obtain $y = z(0)e^{gt}l$, where $z(0)$ is the initial value of z in the steady state. Using these equations, Eq. (A.16) can be rewritten as follows:

$$\lim_{t\to\infty}\gamma\pi_i z(0)le^{-\rho t}=0. \tag{A.21}$$

In the steady state, $\gamma\pi_i z(0)l$ is constant and bounded, and therefore, Eq. (A.16) holds.

A.3 Household optimization problem

Household j's optimization problem can be summarized as follows:

$$\begin{aligned}
&\max_{c,m,\omega_j}\ U_j=\int_0^\infty\left[\ln c+\sigma\ln m+\beta\ln G-\nu\frac{l_j^{1+\psi}}{1+\psi}\right]e^{-\rho t}dt,\\
&\text{subject to}\quad l_j=\left(\frac{W_j}{W}\right)^{-\eta}l,\\
&\qquad\qquad\ \dot{a}=ra+w_jl_j+n+v+b-\frac{\delta}{2}\omega_j^2y-c-\tau y-Rm,\\
&\qquad\qquad\ \dot{W}_j=\omega_jW_j.
\end{aligned} \tag{A.22}$$

[21] As index i, which indicates a variety of goods (or firms), is normalized at 1 here (see Eq. 1), variable y represents not only the amount of the individual good produced but also that of the composite good produced. The connotations of variables p and π are similar.

The Hamiltonian function of this problem is defined as follows:

$$\mathcal{H} = \ln c + \sigma \ln m - \frac{\nu}{1+\psi}\left\{\left(\frac{W_j}{W}\right)^{-\eta} l\right\}^{1+\psi}$$
$$+ \mu_1\{ra + \frac{W_j}{p}\left(\frac{W_j}{W}\right)^{-\eta} l + n + v + b - \frac{\delta}{2}\omega_j^2 y - c - \tau y - Rm\} + \mu_2 \omega_j W_j, \tag{A.23}$$

where μ_1 and μ_2 are the co-state variables associated with the state variables a and W_j, respectively. The first-order conditions for optimality are given by

$$\frac{\partial \mathcal{H}}{\partial c} = \frac{1}{c} - \mu_1 = 0, \tag{A.24}$$
$$\frac{\partial \mathcal{H}}{\partial m} = \frac{\sigma}{m} - \mu_1 R = 0, \tag{A.25}$$
$$\frac{\partial \mathcal{H}}{\partial \omega_j} = -\mu_1 \delta \omega_j y + \mu_2 W_j = 0, \tag{A.26}$$
$$\dot{\mu}_1 = \rho\mu_1 - \frac{\partial \mathcal{H}}{\partial a} = (\rho - r)\mu_1, \tag{A.27}$$
$$\dot{\mu}_2 = \rho\mu_2 - \frac{\partial \mathcal{H}}{\partial W_j} = \rho\mu_2 - \{\frac{\nu\eta l_j^{1+\psi}}{W_j} + \mu_1(1-\eta)\frac{l_j}{p} + \mu_2\omega_j\}. \tag{A.28}$$

The transversality conditions are given by

$$\lim_{t\to\infty} \mu_1 a e^{-\rho t} = 0, \tag{A.29}$$
$$\lim_{t\to\infty} \mu_2 W_j e^{-\rho t} = 0. \tag{A.30}$$

Eqs. (A.24)–(A.26) are the first-order conditions for maximizing $\mathcal{H}$. The Hessian of $\mathcal{H}(c, m, \omega_j)$ is given by

$$|H| = \begin{vmatrix} -\frac{1}{c^2} & 0 & 0 \\ 0 & -\frac{\sigma}{m^2} & 0 \\ 0 & 0 & -\mu_1\delta y \end{vmatrix}. \tag{A.31}$$

The principal minors of $|H|$ (which we denote $|H_1|$, $|H_2|$, and $|H_3|$) are given by

$$|H_1| = -\frac{1}{c^2},\ |H_2| = \begin{vmatrix} -\frac{1}{c^2} & 0 \\ 0 & -\frac{\sigma}{m^2} \end{vmatrix},\ |H_3| = |H|. \tag{A.32}$$

As $|H_1| < 0$, $|H_2| > 0$, and $|H_3| < 0$, the second-order conditions for maximizing $\mathcal{H}$ are satisfied.

We derive Eqs. (19) and (20) from Eqs. (A.24)–(A.28).

Derivation of Eq. (19)

Taking the logarithmic derivative of Eq. (A.24), we obtain

$$\frac{\dot{\mu}_1}{\mu_1} = -\frac{\dot{c}}{c}. \tag{A.33}$$

Substituting this equation into Eq. (A.27) yields

$$-\frac{\dot{c}}{c} = \rho - r. \tag{A.34}$$

Using $r = R - \pi$, Eq. (A.34) can be rewritten as follows:

$$\frac{\dot{c}}{c} + \pi + \rho = R. \tag{A.35}$$

We can eliminate μ_1 from Eq.(A.25) using Eq. (A.24) to obtain

$$R = \sigma\frac{c}{m}. \tag{A.36}$$

Combining Eqs. (A.35) and (A.36) will yield the following:

$$\frac{\dot{c}}{c} + \pi + \rho = R = \sigma\frac{c}{m}. \tag{A.37}$$

Derivation of Eq. (20)

Dividing both sides of Eq. (A.28) by μ_2, we obtain

$$\frac{\dot{\mu}_2}{\mu_2} = \rho - \{\frac{\nu\eta l_j^{1+\psi}}{W_j} + \mu_1(1-\eta)\frac{l_j}{p}\}/\mu_2 - \omega_j. \tag{A.38}$$

We also obtain $\mu_1 = \frac{1}{c}$ from Eq. (A.24), and $\mu_2 = \frac{\delta\omega_j y}{cW_j}$ and $\frac{\dot{\mu}_2}{\mu_2} = \frac{\dot{\omega}_j}{\omega_j} + \frac{\dot{y}}{y} - \frac{\dot{c}}{c} - \frac{\dot{W}_j}{W_j}$ from Eq. (A.26). Substituting these equations into Eq. (A.38) yields the following:

$$\frac{\dot{\omega}_j}{\omega_j} + \frac{\dot{y}}{y} - \frac{\dot{c}}{c} - \omega_j = \rho - \{\frac{\nu\eta l_j^{1+\psi}}{W_j} + (1-\eta)\frac{l_j}{pc}\}\frac{cW_j}{\delta\omega_j y} - \omega_j. \tag{A.39}$$

As all households behave similarly in accordance with Eq. (A.39), we can drop subscript j from W_j, ω_j, and l_j; that is, $W_j = W$, $\omega_j = \omega$, and $l_j = l$. Taking these equations into account, Eq. (A.39) can be rewritten as follows:

$$\frac{\dot{\omega}}{\omega} = \rho + \frac{\dot{c}}{c} - \frac{\dot{y}}{y} - \frac{\nu\eta}{\delta}l^{1+\psi}\frac{c}{\omega y} + \frac{\eta-1}{\delta}\frac{w}{z}\frac{1}{\omega}. \tag{A.40}$$

Transversality conditions

We show that the steady state obtained in Section 3 satisfies the transversality conditions given by Eqs. (A.29) and (A.30).

(1) Transversality condition (A.29)

Using Eq. (A.24), $\mu_1 a$ can be rewritten as follows:

$$\mu_1 a = \frac{a}{c} = \frac{A}{p}\frac{1}{c} = \frac{M+Q}{p}\frac{1}{c}. \tag{A.41}$$

If $\frac{M+Q}{p}\frac{1}{c}$ is constant and bounded in the steady state, Eq. (A.29) will be satisfied.

Dropping subscript i from Eq. (11) yields the following:

$$\begin{aligned}\Pi &= y - wl - \frac{\gamma}{2}\pi^2 y \\ &= (1 - s - \frac{\gamma}{2}\pi^2)zl,\end{aligned} \tag{A.42}$$

where $s = \frac{w}{z}$.

As the firms' profits are distributed to dividends on stock, the stock price $\frac{Q}{p}$ must be equivalent to the discounted present value of a steam of dividends (where the discount rate $r = \rho + g$):

$$\begin{aligned}\frac{Q}{p} &= \int_t^\infty \Pi(t)e^{-(\rho+g)(s-t)}ds \\ &= \left[-\frac{1}{\rho+g}\Pi(t)e^{-(\rho+g)(s-t)}\right]_t^\infty \\ &= \frac{1}{\rho+g}\Pi(t).\end{aligned} \tag{A.43}$$

Substituting Eq. (A.42) into Eq. (A.43), we obtain

$$\frac{Q}{p} = \frac{1}{\rho+g}(1 - s - \frac{\gamma}{2}\pi^2)zl. \tag{A.44}$$

In addition, substituting Eq. (A.24) into Eq. (A.25), we obtain

$$\frac{M}{p} = \sigma\frac{c}{R}. \tag{A.45}$$

Eqs. (9), (15), and (23) are combined to give the following:

$$c = (1 - \tau)zl. \tag{A.46}$$

We can substitute Eqs. (A.44), (A.45), and (A.46) into the rightmost side of Eq. (A.41) to obtain

$$\frac{M+Q}{p}\frac{1}{c} = \frac{1 - s - \frac{\gamma}{2}\pi^2 + \frac{\sigma(1-\tau)}{R}}{(\rho + g)(1-\tau)}, \tag{A.47}$$

which is constant and bounded in the steady state, and therefore, the transversality condition given by Eq. (A.29) is satisfied.

(2) Transversality condition (A.30)

Eq. (A.26) demonstrates the following:

$$\begin{aligned} \mu_2 W_j &= \mu_1 \delta\omega_j y \\ &= \delta\omega_j l \frac{y}{c} \\ &= \delta\omega_j l \frac{zl}{(1-\tau)zl} \\ &= \frac{\delta\omega_j}{1-\tau}, \end{aligned} \tag{A.48}$$

which is constant and bounded in the steady state, and therefore, the transversality condition given by Eq. (A.30) is satisfied.

A.4 Derivation of Eqs. (24)–(28)

In this section, we derive differential equations (24)–(28).

Derivation of Eq. (24)

The logarithmic derivative of Eq.(19) is given by

$$\frac{\dot{R}}{R} = \frac{\dot{c}}{c} - \frac{\dot{m}}{m}. \tag{A.49}$$

Using Eq. (19), we can write the growth rate of c as follows:

$$\frac{\dot{c}}{c} = R - \pi - \rho. \tag{A.50}$$

Substituting Eqs. (A.50) and (22) into Eq. (A.49), we obtain

$$\dot{R} = (R - \theta - \rho)R. \tag{A.51}$$

Derivation of Eq. (25)

The logarithmic derivative of Eq. (9) is given by

$$\frac{\dot{y}}{y} = g + \frac{\dot{l}}{l}. \tag{A.52}$$

Using Eqs. (15) and (23), we can obtain $(1-\tau)y = c$. Taking the logarithmic derivative of this equation, we obtain

$$\frac{\dot{y}}{y} = \frac{\dot{c}}{c}. \tag{A.53}$$

Substituting this equation into Eq. (A.52), we obtain

$$\frac{\dot{c}}{c} = g + \frac{\dot{l}}{l}. \tag{A.54}$$

Substituting Eq. (A.50) into Eq. (A.54) will yield the following:

$$\dot{l} = (R - \rho - \pi - g)l. \tag{A.55}$$

Derivation of Eq. (26)

Using the relation $r = R - \pi$, Eq. (19) can be rewritten as follows:

$$\frac{\dot{c}}{c} = r - \rho. \tag{A.56}$$

Substituting Eq. (A.56) into Eq. (13) using $\frac{\dot{y}}{y} = \frac{\dot{c}}{c}$, we obtain

$$\dot{\pi} = \rho\pi + \frac{\phi - 1}{\gamma} - \frac{\phi}{\gamma}s, \tag{A.57}$$

where $s = \frac{w}{z}$.

Derivation of Eq. (27)

From Eqs. (15) and (23), we obtain $(1-\tau)y = c$ and $\frac{\dot{y}}{y} = \frac{\dot{c}}{c}$. Substituting these equations into Eq. (20) and defining $s = \frac{w}{z}$, we obtain

$$\frac{\dot{\omega}}{\omega} = \rho - \frac{\nu\eta}{\delta}l^{1+\psi}\frac{1-\tau}{\omega} + \frac{\eta - 1}{\delta}s\frac{1}{\omega}. \tag{A.58}$$

Multiplying both sides of this equation by ω will yield the following:

$$\dot{\omega} = \rho\omega - \frac{\nu\eta}{\delta}l^{1+\psi}(1-\tau) + \frac{\eta - 1}{\delta}s. \tag{A.59}$$

Derivation of Eq. (28)

The logarithmic derivative of $s = \frac{w}{z}$ is given by

$$\frac{\dot{s}}{s} = \frac{\dot{w}}{w} - g. \tag{A.60}$$

Substituting $\frac{\dot{w}}{w} = \omega - \pi$ into Eq. (A.60) and multiplying both sides of it by s, we obtain

$$\dot{s} = (\omega - \pi - g)s. \tag{A.61}$$

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