SHOULD THE SOUTH AFRICAN RESERVE BANK RESPOND TO EXCHANGE RATE FLUCTUATIONS? EVIDENCE FROM THE COSINE-SQUARED CEPSTRUM

ABSTRACT

Empirical evidence on the whether the inflation-targeting South African Reserve Bank (SARB) should also consider responding to exchange rate fluctuations, are contradictory. Against this backdrop of contradictory evidence, we revisit the issue by questioning if the inflation rate is more volatile *than it would have been* had South Africa not moved to a flexible exchange rate regime in 1995, using the *cosine-squared cepstrum*. We find that the CPI inflation in South Africa has become more volatile since the second quarter of 1995, post a flexible exchange rate regime, than it would have been had the country continued to pursue a fixed exchange rate policy. Based on this result, we can conclude that the SARB should perhaps respond to exchange rate fluctuations, however, we also warn against the cost of increased volatility in output that is likely to result from targeting exchange rate variability.

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1. INTRODUCTION

In the February of 2000, the Minister of Finance announced that inflation targeting would be the sole objective of the South African Reserve Bank (SARB). Currently, the Reserve Bank's main monetary policy objective is to maintain CPIX[1] inflation between the target-band of three to six percent, using discretionary changes in the repo rate as its main policy instrument. Given that, nominal currency depreciation feeds into domestic inflation directly by increasing the foreign component of CPI, and indirectly through its effect on the marginal costs of domestic producers, a pertinent question for an inflation targeting country like South Africa is: Should the SARB condition on exchange rate movements when it sets its interest rate policy? Or in other words, should the SARB target the exchange rate?[2]

Two recent studies on South Africa tend to provide diametrically opposite answers. On one hand, Alpanda *et al*., (2010), based on a New Keynesian small open economy Dynamic Stochastic General Equilibrium (DSGE) model, indicates that the optimal Taylor rule (derived from minimization of a loss function subject to the structure of the economy) responds to inflation and output, but almost no weight is placed on the depreciation of currency. On the other, Naraidoo and Raputsoane (2010), also using a New Keynesian model, but allowing for the objective function of the SARB to account

[1] CPIX is defined as CPI excluding interest rates on mortgage bonds.

[2] Current currency depreciation could also provide additional information to the central bank regarding current CPI inflation, since the central bank observes inflation rates only with a lag. This informational aspect also supports the use of an interest rate rule that responds to currency depreciation.

for loss due to the deviation of a financial conditions index from its long-run path, show that the central bank pay close attention to financial conditions index when settings its repo rate.[3] Though Naraidoo and Raputsoane (2010) did not explicitly consider the exchange rate deviations in the loss function, their financial conditions index was constructed as a weighted average of the real effective exchange rate, the real house price index, the real stock price, the credit spread and the future spread.

Against this backdrop of contradictory evidence, we revisit the issue by reformulating the question to: Is the inflation rate more volatile *than it would have been* had South Africa not moved to a flexible exchange rate regime in 1995?[4] To address this issue, we use what is called the *cosine-squared cepstrum*, and apply the technique to analyze the Consumer Price Index (CPI) inflation volatility in South Africa over the period of 1991:Q2 to 2011:Q2, with the starting point determined by the methodology used and the endpoint by data availability at the time of writing this paper. Note that we use quarterly data on CPI and not CPIX inflation, since the latter series is not available beyond 2002:Q2 and our methodology requires data dating back to 1991:Q2.[5] Understandably, if moving to a flexible exchange rate regime had increased the volatility of inflation than it would have otherwise been if the exchange rate was fixed, then it could be suggesting to us that the SARB should target the exchange rate.

Given that the exchange rate regime is now a flexible one, if the move into the new regime or the abandonment of the older one affected inflation volatility, then observed inflation is represented as the sum of two series: (i) the series that would have eventuated if the exchange rate was fixed, and (ii) a second series associated with the direct impact of the regime change that arrived in 1995. If the regime had any coherent effect, then the two series are likely to be well correlated. If the second series is found to be positively correlated with the series that would have eventuated, then it increases the volatility of inflation and is said to be in-phase. However, if the second series is negatively correlated or out of phase, volatility declines, since fluctuations in the series that would have eventuated are dampened. Generally speaking, if a time-varying stationary series is composed of two such series that are linear or well correlated, then the autocovariance function contains a global maximum at the zero lag and a local extremum at a lag corresponding to the date when the second series arrives. The problem with the autocovariance function is that it is difficult to detect a second series and determine the degree of its phase shift, given that a local extremum, in the case of an autocovariance function, appears as a broad cycle.

The cepstrum technique helps us to overcome such difficulties. Cepstra have been used successfully in detecting secondary influences in engineering, particularly communication theory. Intuitively, the cosine-squared cepstrum behaves like an autocovariance function, but with sharper resolution that helps in identifying the arrival and phase relationship of the secondary series with great precision. A local extremum appears as an impulse, instead of a broad cycle, the direction of which, in turn, determines whether the secondary series have increased or dampened volatility. Because of this, the

[3] Similar results were also obtained, both for in-sample and out-of-sample analyses, by Naraidoo and Kasai (forthcoming) and Kasai and Naraidoo (2011) based on single-equation linear and non-linear Taylor-type rules augmented with the financial conditions index.

[4] For a detailed discussion on the history of the exchange rate regimes in South Africa, refer to Ocran (2010).

[5] For further details, refer to Section 3 below.

cosine-squared cepstrum is well equipped to determine whether inflation volatility is greater than it would otherwise have been if had the SARB continued to pursue a fixed exchange rate policy. To the best of our knowledge, this is the first attempt to study inflation volatility across exchange rate regimes by using the cosine-squared cepstrum. The two other papers that have used the cosine-squared cesptrum in economics is by Cunnigham and Vilasuso (1994) and Gupta and Uwilingiye (forthcoming). While, Cunningham and Vilasuso (1994) used the technique to compare Gross National Product (GNP) volatility across exchange rate regimes for the US economy, Gupta and Uwilingiye (forthcoming) used the same to analyze whether moving to an inflation targeting regime in South Africa had increases the volatility of CPI inflation than it would have otherwise been had the SARB continued to pursue the traditional more eclectic approach to monetary policy. The remainder of the paper is organized as follows: Section 2 describes the technical details of a cepstrum, while, the data and our main findings are discussed in Section 3. Finally, Section 4 concludes.

2. THE CEPSTRUM[6]

Speaking mathematically, the cepstrum is an integral transform with a long history. The use of the cepstrum dates back to Poisson (1823) and Schwarz (1872), who applied cepstra to problems involving potential functions with real parts fixed on the unit circle. Later, Szëgo (1915) and Kolmogorov (1939) used cepstra in the extraction of stable causal systems by factoring power spectra of random processes. However, it is the application of Bogert et al. (1963) in engineering that most coincides with our interest here.

Bogert et al. (1963) considers a time-varying function *f(t)* which is made up of another function $f_1(t)$ and its additive "echo," $f_1(t-\tau)$, lagged by τ periods. Formally, we have:

$$f(t) = f_1(t) + af_1(t-\tau) \tag{1}$$

The power spectrum of $f(t)$ is

$$|F(\omega)|^2 = |F_1(\omega)|^2 \left\{1 + a^2 + 2a\cos(\omega\tau)\right\}, \tag{2}$$

where $F_1(w)$ is the complex Fourier transform of $f_1(t)$. The "echo", in turn, manifests as a cosine function riding on the envelope of the power spectrum. The period of the cosine function is the reciprocal of the lag τ. In studies involving autocorrelation analysis in hydroacoustic problems, Griffin et al. (1980) has shown that if the spectra of $f(t)$ and its "echo" differ, i.e., they are not perfectly correlated, then the parameter a in Equation (2) varies with frequency. Thus, $f(t)$ is better represented by:

$$f(t) = f_1(t) + f_2(t-\tau), \qquad f_1(t) \neq f_2(t) \tag{3}$$

[6] This section relies heavily on the discussion available in Cunnigham and Vilasuso (1994) and Gupta and Uwilingiye (forthcoming).

so that the power spectrum of $f(t)$ is given by:

$$|F(\omega)|^2 = |F_1(\omega)|^2 + |F_2(\omega)|^2 + 2|F_1(\omega)||F_2(\omega)|\cos\{\phi_1(\omega) - \phi_2(\omega) + \omega\tau\}, \tag{4}$$

where $\phi_1(\omega)$ and $\phi_2(\omega)$ are the phase spectra of $f_1(t)$ and $f_2(t)$, respectively.

The modulating cosine is phase-shifted by an amount which is equal to the differences in the phases of functions that composes $f(t)$. Thus, if the function $f_1(t)$ and $f_2(t)$ are not close copies of one another, the argument of the modulating cosine is not constant, which, in turn, has important consequences as we shall see below.

We proceed under the assumption that $f_1(t) = f_2(t)$ and return to the more general case later. The function $f(t)$ can be represented as the linear convolution of $f_1(t)$ with a train of impulses. Bogert et al. (1963) argue that if the envelope of $|F(\omega)|^2$ could be made optimally white, this would be equivalent to making $|F_1(\omega)|^2$ into a "boxcar" function, whose Fourier transform would be a sinc function at the origin.[7] In the limit, as the envelope of the power spectrum becomes uniform at all frequencies, the sinc function tends towards a Dirac delta function. The transform of the modulating cosine would be an impulse whose delay is related to the frequency of the modulating cosine which, in turn, is equal to the lag length between $f_1(t)$ and its "echo." Therefore, under ideal conditions and when scaled properly, this resulting series yields a time domain function with a global maximum, or "peak" at the origin, and the local maximum or "peak" indicating the arrival time of the "echo."

In practice, the whitening of the power spectrum is performed by first applying the natural logarithm and then the inverse Fourier transform (IFT). Because it ignores the phase spectrum, and is calculated directly from the log power spectrum, the resultant function is referred to as the *power spectrum*.

Let us now go back to case presented in Equations (3) and (4). If the component functions are not close copies of one another, then the argument of the modulating cosine is not invariant, and, hence, cepstral peaks rapidly degenerate. Therefore, it must be realized that the impulse appears at the appropriate lag of the cepstrum *only if the component functions are well correlated.*

When we consider the discrete case, cepstra are a class of integral transform whose kernel is a function of the *z*-transform of a real sequence. The discrete power cepstrum of a data sequence $x(nT)$ with *z*-transform $X(z)$ is then given by:

$$\tilde{x}(nT) = \frac{1}{2\pi i}\left|\oint_c \log|X(z)|\, z^{n-1} dz\right|^2 \tag{5}$$

where $n = 0, 1, 2, \ldots, N$, enumerates the samples, T is the sampling interval, and C is a closed contour inside the region of convergence of the power series and enclosing the origin. As discussed by Cunningham (1980), this can be extended to the *cosine-squared cepstrum* by addition of the signum function as follows:

[7] Note that *sinc x = (sinx)/x*.

$$\hat{x}(nT) = \tilde{x} \times \operatorname{sgn}\left\{ \operatorname{Re}\left[\frac{1}{2\pi i} \oint_c \log \left| X(z) \right| z^{n-1} dz \right] \right\} \tag{6}$$

The addition of the signum function allows the cepstrum to determine not only the arrival time of the secondary series, but also, and perhaps more importantly, its polarity relative to the original series making its interpretation analogous to the autocovariance function.[8] Because in real-life applications the real sequences $x(nT)$ are of finite length, the annulus of convergence of $X(z)$ always includes the unit circle, the transforms may be computed by means of the fast Fourier transforms.

Because the cepstrum is essentially the spectrum of a spectrum, the cepstral domain is a time domain. The terminology easily becomes confusing, therefore, Bogert et al. (1963) suggested the following conventions: the term "cepstrum" is an anagram of the word "spectrum." Likewise, periodicities in the cepstrum are discussed in terms of "quefrencies," "gamnitudes," and "repiods," analogous to "frequencies," "gain/amplitudes," and "periods" in the time domain. "Filtering" in the cepstral domain is "liftering" and so on. Further, Bogert *et al.* (1963) refers to data analysis in the cepstral domain as "alanysis". Finally, the term "saphe," pronounced "safe" and related to "phase," is used to refer to the displacement between the secondary, or lagged series, and the original. Thus, the detection and analysis of cepstral peaks and "phase" shifts of the lagged series is referred to as "saphe cracking."

3. APPLICATION TO INFLATION VOLATILITY

If the move to the flexible exchange rate regime affected the volatility of the CPI inflation rate for South Africa, then the series is representable as the sum of the pre-1995 series and a secondary series arriving 1995. If the secondary series are perfectly in-phase with the pre-1995 series, then volatility increased relative to what it would have been under the fixed exchange rate regime; if it is perfectly out-of-phase, then volatility decreased. Determining whether or not a secondary series arrived and it was in-phase or out-of-phase is a straightforward application of the cosine-squared cepstrum, discussed in Section 2.

We begin by performing the nonparametric Phillips and Perron (1988) [PP] ϱ-test for unit root tests on the seasonally adjusted annualized CPI inflation rate,[9] since the application of the cosine-squared cepstrum requires us to ensure that the data process is stationary. Note that the PP test has a null of non-stationarity. The PP statistic is computed with a truncation parameter $p=[CT]^k$ in the Newey-West variance estimator, where c=5 and k= 0:2 are adopted following Bierens (1997). Since the PP tests may be subject to size distortion in finite samples, its p-values are simulated on the basis of

[8] The close relationship between the cosine-squared cepstrum and the autocovariance function becomes clear when we write the latter as a function of the z-transform of the same series: $x_{acvf}(\tau) = \frac{1}{2\pi i} \oint_c \left| X(z) \right| z^{n-1} dz.$ Equivalently, the autocovariance function is the inverse Fourier transform of the power spectrum of a series.

[9] The data for this paper are from the International Monetary Fund's International Financial Statistics data base. The computations involved in obtaining the cepstrum are performed by using the Signal Processing Toolbox in MATLAB, Version R 2009a.

1000 replications of a Gaussian AR(1) process for the underlying variables in first differences. As Table 1 reports, the test results validate that the inflation rate is an I(1) processes. The series is, however, found to be stationary in its first differences; hence CPI inflation is integrated of order 1. As the variable of interest is found to be non-stationary, we had to first difference the inflation rate series. Once the first-differenced CPI inflation series was generated and found to be stationary, we demean the series to avoid the dominance of the zero frequency components in the power spectrum, and then apply the five point Hanning-type cosine tapers to suppress possible sidelobes.

Table 1. Unit root test results for the CPI Inflation

Series	$Z(t)^{a}$	$Z(t_{\mu})^{b}$	$Z(t_{\tau})^{c}$
Inflation	-1.622**(0.099)	-2.278(0.182)	-2.745(0.222)
Δ(Inflation)	-5.432*(0.000)	-5.424*(0.000)	-5.415*(0.000)

Notes: Inflation stands for CPI inflation and Δ(Inflation) stands for the first-differenced CPI inflation.

*, **, *** indicate significance at the 1%, 5%, and 10% levels, respectively.

[a] Test allows for neither a constant nor a trend; One-sided test of the null hypothesis that the variable is nonstationary; 1%, 5%, and 10% critical values equal -2.594, -1.945, and -1.614, respectively.

[b] Test allows for a constant; One-sided test of the null hypothesis that the variable is nonstationary; 1%, 5%, and 10% critical values equal -3.513, -2.898, and -2.586, respectively.

[c] Test allows for a constant and a linear trend; One-sided test of the null hypothesis that the variable is nonstationary; 1%, 5%, and 10% critical values equal -4.075, -3.466, and -3.160, respectively.

Simulated *p*-values in parentheses.

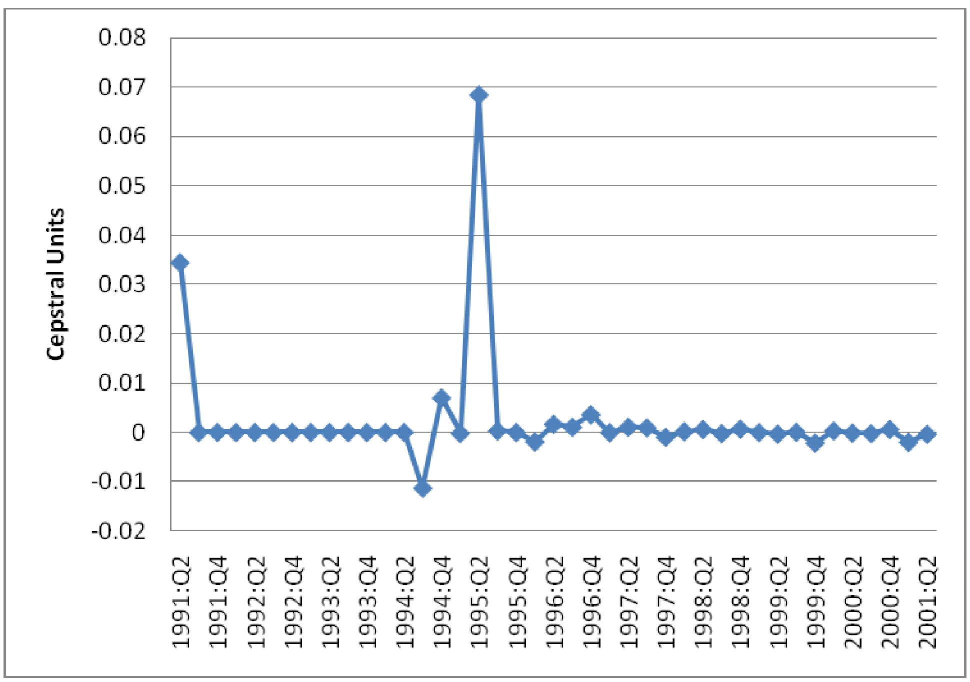
Next we use a Cooley–Tukey fast Fourier transform (FFT) and compute the natural logarithm of the sum of squares of the real and imaginary parts to form the log power spectrum. Note we start our analysis 15 quarters prior to the change in regime, so the first-differenced CPI inflation series begins at 1991:Q2. The choice of 15 quarters is decided based on the average distance between the troughs of the smoothed[10] log power spectrum, which was found to be approximately 0.0686 cycles per quarter, so that the lag associated with the secondary influence on the first-differenced inflation rate is approximately at a lag of 15 quarters, i.e., $\tau = 15$, arriving in the first quarter of 1995. Given that the seasonally (adjusted at annual rate) CPIX inflation rate was also

[10] Following Bogert *et al.* (1963), the power spectrum is smoothed by applying a five-point centered moving average, since the moving average suppresses high frequency variations that would correspond to relatively long lags in secondaries.

non-stationary, using the first-differenced CPIX inflation rate over 2002:Q3-2010:Q3 revealed that the distance between the troughs of the smoothed log-power spectrum was approximately 0.06812 cycles per quarter on average, implying a $\tau = 15$ as well. With CPIX inflation data only available from 2002:Q2 and the regime change taking place in 1995:Q1, we could not use the CPIX inflation data to obtain the cosine-squared cepstrum. Forming the inverse FFT (IFFT) and then the sum of squares again, and establishing the sign according to the sign of the real part, the cosine-squared cepstrum is formed, as discussed in Equation (6) in Section 2.

Figure 1 shows the first 41 lags of the cosine-squared cepstrum for first-differenced CPI inflation. For the sake of convenience, the time scale replaces the lag numbers. The first twelve points of the cepstrum for the first-differenced CPI inflation rate is rescaled to enhance the detail.[11] A delta function spikes prominently upward at the lag corresponding to the second quarter of 1995. This "arrival time" for the secondary series is consistent with the move to a flexible exchange rate regime in 1995. The unambiguous nature of the delta function and its positive sign provide evidence of a secondary influence on the South African CPI inflation series that not only matched it fluctuation by fluctuation, but was also exactly in-phases with the CPI inflation series. In other words, the secondary influence that arrived in 1995:Q2 has increased the fluctuations/volatility in the CPI inflation rate.

FIGURE 1. THE COSINE-SQUARED CEPSTRUM OF THE CPI INFLATION, (1991:Q2-2011:Q2)

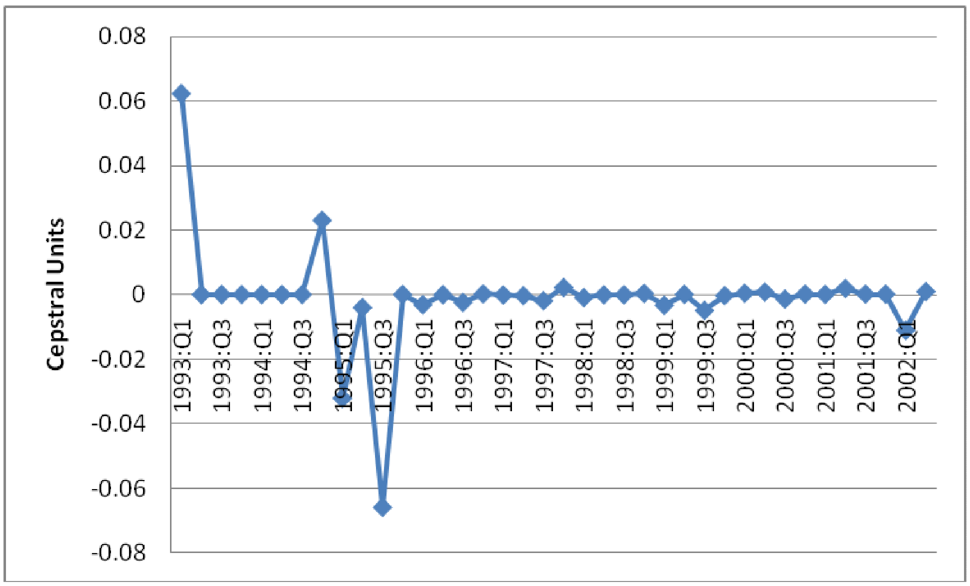


[11] The first twelve cepstral estimates for the first-differenced CPI inflation rate is set equal to zero to enhance the detail. The above operation removes the delta function at the origin, which, in turn corresponds to the global maximum.

4. CONCLUSIONS

The positive delta function in the cosine-squared cepstrum has provided evidence that the flexible exchange rate regime in South Africa began to impact CPI inflation in the second quarter of 1995, making the same more volatile than it would have been had the exchange rate regime continued to be a fixed one. The cosine-squared cepstrum, thus, shows that it can be successfully used in applications where the effects policy changes on time series data can be modelled as additive time series that are well correlated with the variable under study. However, there are limitations to this approach: First, the increased inflation volatility under the new exchange rate regime may not be permanent, but rather a pulse-like response in the inflation rate, and; Second, at times the methodology might require additional economic insight to isolate the possible economic causes of the event, without recovering the secondary series. Fortunately in our case, the prominent cepstral peak is relatively unambiguous in terms of its timing and phase characteristics.

FIGURE 2. THE COSINE-SQUARED CEPSTRUM OF THE GDP GROWTH RATE, (1993:Q1-2011:Q2)



Despite the limitations, the finding of increased inflation volatility in the flexible exchange rate regime indicated by the positive delta function in the cosine-squared cepstrum cannot be taken lightly. The result tend to suggest that the SARB should target the exchange rate to reduce volatility in the CPI inflation, given that its sole objective since adopting an inflation targeting framework since the February of 2000, is price stability.[12] Having said that, this in turn, is likely to increase the variability of output, especially when currency depreciation rates are volatile. And in fact, when we

[12] It must be realized that for a country seeking price stability, it is not only essential to obtain lower mean levels of inflation but also less volatility in inflation. Inflation volatility matters because high variability of inflation over time makes expectations about the future price level more uncertain, which, in a world with nominal contracts, induces risk premia for long-term arrangements, raises costs for hedging against inflation risks and leads to unanticipated redistribution of wealth. Thus, inflation volatility can impede growth, even if inflation on average remains restrained.

analyzed the cosine-squared cepstrum of the growth rate of the Gross Domestic Product (GDP), we find a dominant negative delta function in 1995:Q3, suggesting that the flexible exchange rate has reduced the volatility of output than it would otherwise have been if the fixed exchange rate regime continued. Figure 2 presents the cosine-squared cepstrum for the growth rate of the GDP based on a $\tau = 8$, since the distance between the troughs of the smoothed log-power spectrum was approximately 0.1250 cycles per quarter on average. Thus, the answer to whether the SARB should respond to exchange rate fluctuations or not, is not that clear after all, since it would depend on whether the SARB weighs the fluctuations on inflation more than that of the output. However, when we look at the available evidence provided in Ortiz and Sturzenegger (2007), Naraidoo and Gupta (2010), Alpanda *et al.,* (2010), Naraidoo and Kasai (forthcoming), Kasai and Naraidoo (2011) and Naraidoo and Raputsoane (2010, 2011) amongst others, historically, the SARB is found to respond more strongly to the deviation of inflation rate from the target than the output gap. In light of this result, and the evidence provided by the cosine-squared cepstrum, perhaps, the SARB should then consider targeting exchange rate fluctuations. And if we are to believe in the evidence provided by Naraidoo and Raputsoane (2010), Naraidoo and Kasai (forthcoming) and Kasai and Naraidoo (2011), then the SARB has been doing so by targeting a financial conditions index, which involves the real effective exchange rate.

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